

# SUPERTROPICAL POLYNOMIALS AND RESULTANTS

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**ABSTRACT.** This paper, a continuation of [3], involves a closer study of polynomials of supertropical semirings and their version of tropical geometry in which we introduce the concept of relatively prime polynomials and resultants, with the aid of some topology. Polynomials in one indeterminant are seen to be relatively prime iff they do not have a common tangible root, iff their resultant is tangible. The Frobenius property yields a morphism of supertropical varieties; this leads to a supertropical version of Bézout's theorem. Also, a supertropical variant of factorization is introduced which yields a more comprehensive version of Hilbert's Nullstellensatz than the one given in [3].

## 1. INTRODUCTION AND REVIEW

The supertropical algebra, a cover of the max-plus algebra, explored in [2], [3], was designed to provide a more comprehensive algebraic theory underlying tropical geometry. The abstract foundations of supertropical algebra, including polynomials over supertropical semifields, are given in [3]. The corresponding matrix theory is explored in [4], and this paper is a continuation, exploring the resultant of supertropical polynomials in terms of matrices, and the ensuing applications to the resultant. The tropical resultant has already been studied by Sturmfels [5, 6], Dickenstein, Feichtner, and Sturmfels [1], and Tabera [7], but our purely algebraic approach is quite different, leading to a tropical version of Bézout's Theorem (Theorem 5.1).

Since this paper deals mainly with polynomials and their roots, it could be viewed as a continuation of [3], although we explicitly state those results that we need. We briefly review the underlying notions. The underlying structure is a **semiring with ghosts**, which we recall is a triple  $(R, \mathcal{G}, \nu)$ , where  $R$  is a semiring with zero element  $0_R$  (often identified in the examples with  $-\infty$ , as indicated below), and  $\mathcal{G}_0 = \mathcal{G} \cup \{0_R\}$  is a semiring ideal, called the **ghost ideal**, together with an idempotent semiring homomorphism

$$\nu : R \longrightarrow \mathcal{G} \cup \{0_R\}$$

called the **ghost map**, i.e., which preserves multiplication as well as addition. We write  $a^\nu$  for  $\nu(a)$ , called the  $\nu$ -**value** of  $a$ . Two elements  $a$  and  $b$  in  $R$  are said to be  $\nu$ -**matched** if they have the same  $\nu$ -value; we say that  $a$  **dominates**  $b$  if  $a^\nu \geq b^\nu$ . Two vectors are  $\nu$ -**matched** if their corresponding entries are  $\nu$ -matched.

**Note 1.1.** *Throughout this paper, we also assume the key property called **supertropicality**:*

$$a + b = a^\nu \quad \text{if} \quad a^\nu = b^\nu.$$

*In particular,  $a + a = a^\nu$ ,  $\forall a \in R$ .*

A **supertropical semiring** has the extra structure that  $\mathcal{G}$  is ordered, and satisfies the property called **bipotency**:  $a + b = a$  whenever  $a^\nu > b^\nu$ .

A **supertropical domain** is a supertropical semiring for which  $\mathcal{T}(R) = R \setminus \mathcal{G}_0$  is a monoid, called the set of **tangible elements** (denoted as  $\mathcal{T}$  when  $R$  is unambiguous), such that the map  $\nu_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$  (defined as the restriction from  $\nu$  to  $\mathcal{T}$ ) is onto. We write  $\mathcal{T}_0$  for  $\mathcal{T} \cup \{0_R\}$ . We also define a **supertropical**

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**semifield** to be a commutative supertropical domain  $(R, \mathcal{G}, \nu)$  for which  $\mathcal{T}$  is a group; in other words, every tangible element of  $R$  is invertible. Thus,  $\mathcal{G}$  is also a (multiplicative) group. Since any strictly ordered commutative semigroup has an ordered Abelian group of fractions, one can often reduce from the case of a (commutative) supertropical domain to that of a supertropical semifield.

When studying a supertropical domain  $R$ , it is convenient to define an inverse function  $\hat{\nu} : R \rightarrow \mathcal{T}$ , which is a retract of  $\nu$  in the sense that  $\hat{\nu}$  is  $1_{\mathcal{T}_0}$  on  $\mathcal{T}_0$ , and writing  $\hat{a}$  for  $\hat{\nu}(a)$ , we have  $(\hat{a})^\nu = a^\nu$  for any  $a \in R$ . (When  $\nu_{\mathcal{T}}$  is 1:1, we take  $\hat{\nu}$  to be  $\nu_{\mathcal{T}}^{-1}$  on  $\mathcal{G}$ . In general, the function  $\hat{\nu}$  need not be uniquely defined if  $\nu_{\mathcal{T}}$  is not 1:1.)

The following natural topology is very useful in dealing with certain delicate issues.

**Definition 1.2.** For any supertropical domain  $R = (R, \mathcal{G}, \nu)$ , we define the  $\nu$ -**topology** to have a base of open sets of the form

$$W_{\alpha, \beta} = \{a \in R : \alpha^\nu < a^\nu < \beta^\nu\} \quad \text{and} \quad W_{\alpha, \beta; \mathcal{T}} = \{a \in \mathcal{T} : \alpha^\nu < a^\nu < \beta^\nu\}, \quad \text{where} \quad \alpha^\nu, \beta^\nu \in \mathcal{G}_0.$$

We call such sets **open intervals** and **tangible open intervals**, respectively. We say that  $R$  is **connected** if each open interval cannot be written as the union of two nonempty disjoint intervals.

$R^{(n)}$  is endowed with the product topology induced by the  $\nu$ -topology on  $R$ .

**Remark 1.3.**

- (i) Clearly  $a^\nu$  is in the closure of  $\{a\}$ , since any open interval containing  $a^\nu$  also contains  $a$ .
- (ii) The  $\nu$ -topology restricts to a topology on  $\mathcal{T}$ , whose base is the set of tangible intervals.
- (iii) We often will assume that  $R$  (and thus  $\mathcal{T}$ ) is divisibly closed, by passing to the divisible closure  $\{\frac{a}{n} : a \in R, n \in \mathbb{N}\}$ ; see [3, Section 3.4] for details.

**1.1. The function semiring.** Our main connection from supertropical algebra to geometry comes from supertropical functions, which we view in the following supertropical setting:

**Definition 1.4.**  $\text{Fun}(R^{(n)}, R)$  denotes the set of functions from  $R^{(n)}$  to  $R$ . A function  $f \in \text{Fun}(R^{(n)}, R)$  is said to be **ghost** if

$$f(a_1, \dots, a_n) \in \mathcal{G}_0$$

for every  $a_1, \dots, a_n \in R$ ; a function  $f \in \text{Fun}(R^{(n)}, R)$  is called **tangible** if

$$f(J) \not\subseteq \mathcal{G}_0$$

for every nonempty open set  $J$  of  $R^{(n)}$  with respect to the product topology induced by Definition 1.2.

$\text{CFun}(R^{(n)}, R)$  consists of the sub-semiring comprised of functions in the semiring  $\text{Fun}(R^{(n)}, R)$  which are continuous with respect to the  $\nu$ -topology.

**Remark 1.5.**  $\text{Fun}(R^{(n)}, R)$  has the ghost map  $\nu$  given by defining  $f^\nu(a) = f(a)^\nu$ . Thus,  $\text{Fun}(R^{(n)}, R)$  is a semiring with ghosts, satisfying supertropicality, although  $\text{Fun}(R^{(n)}, R)$  is not a supertropical semiring since bipotence fails.

**Remark 1.6.** The product  $fg$  of tangible functions is also tangible. (Indeed, by definition, for any open interval  $W_1$ ,  $f(W_1)$  is not ghost, so therefore  $W_2 = \{a \in W_1 : f(a) \in \mathcal{T}\}$  is a nonempty open set. By definition,  $g(W_2)$  is not ghost, and thus  $fg(W_2)$  is not ghost.)

**Definition 1.7.** Functions  $f_1, \dots, f_m \in \text{CFun}(R^{(n)}, R)$  are  $\nu$ -**distinct** on an open set  $W$  if there is a nonempty dense open set  $W' \subseteq W$  on which  $f_i(\mathbf{a})^\nu \neq f_j(\mathbf{a})^\nu$  for all  $i \neq j$  and all  $\mathbf{a} \in W'$ .

**Remark 1.8.** To satisfy Definition 1.7, it is enough to find dense  $W_{ij} \subseteq W$  for each  $i \neq j$ , such that  $f_i(\mathbf{a})^\nu \neq f_j(\mathbf{a})^\nu$  for all  $\mathbf{a} \in W_{ij}$ , since then one takes  $W' = \bigcap_{i,j} W_{ij}$ .

The idea underlying the definition is that there is a dense subset of  $W$ , at each point of which only one of the  $f_i$  dominates.

**1.2. Polynomials.** Any polynomial can be viewed naturally in  $\text{CFun}(R^{(n)}, R)$ . We say that two polynomials are **e-equivalent** if their images in  $\text{CFun}(R^{(n)}, R)$  are the same; i.e., if they yield the same function from  $R^{(n)}$  to  $R$ . Abusing notation, we sometimes write  $f(\lambda_1, \dots, \lambda_n)$  for a polynomial  $f \in R[\lambda_1, \dots, \lambda_n]$ , indicating that  $f$  involves the variables  $\lambda_1, \dots, \lambda_n$ .

We say that  $f_j$  is **essential** in  $f = \sum_i f_i \in \text{CFun}(R^{(n)}, R)$  if there exists some nonempty open set  $W' \subset R^{(n)}$  for which

$$f_j(\mathbf{a})^\nu > \sum_{i \neq j} f_i(\mathbf{a})^\nu \quad \text{for all } \mathbf{a} = (a_1, \dots, a_n) \in W'.$$

We define the set  $W_{f_j}$  to be the set  $\{\mathbf{a} \in R^{(n)} : f_j(\mathbf{a}) \text{ dominates } (\mathbf{a})\}$ .

The case of an essential monomial of a polynomial, defined in [3], is a special case of this definition. The **essential** part of a polynomial  $f$  is the sum of its essential monomials. Since the essential part of  $f$  has the same image in  $\text{CFun}(R^{(n)}, R)$  as  $f$ , we may assume that the polynomials we examine are essential. Note that a polynomial is ghost (as in Definition 1.4) iff its essential part is a sum of ghost monomials.

**Remark 1.9.** *By definition, for any tangible function, there is a nonempty open set on which it cannot be ghost. Thus, a tangible essential summand of a polynomial  $f$  must dominate at some tangible value.*

Recall that the point  $\mathbf{a} = (a_1, \dots, a_n) \in R^{(n)}$  is called a **root** of a polynomial  $f(\lambda_1, \dots, \lambda_n)$  iff  $f(\mathbf{a})$  is ghost. For  $n = 1$ , we say that a root of  $f(\lambda)$  is **ordinary** if it is a member of an open interval that does not contain any other roots of  $f$ . Likewise, a common root of two polynomials  $f(\lambda)$  and  $g(\lambda)$  is **2-ordinary** if it is a member of an open interval that does not contain any other common roots of  $f$  and  $g$ . (More generally, for  $n > 1$ , a root  $\mathbf{a} \in R^{(n)}$  of  $f$  is said to be **ordinary** if  $\mathbf{a}$  belongs to some open set  $W_{\mathbf{a}}$  which contains a dense subset  $W'$  on which  $f$  is tangible, but we only consider the case  $n = 1$  in this paper.)

**Lemma 1.10.** *Suppose  $f = \sum_i f_i \in \text{CFun}(R^{(n)}, R)$  is ghost on some nonempty open set  $W$  on which the  $f_i$  are  $\nu$ -distinct. Then each summand  $f_j$  of  $f$  that is essential on  $W$  is ghost on an open subset  $W_j$  of  $W$ .*

*Proof.* Otherwise the subset of  $W$  on which  $f_j$  dominates contains a tangible element, and thus contains a tangible open set, contrary to hypothesis.  $\square$

**Remark 1.11.** *We say that a function  $f$  is **tangible at**  $\mathbf{a} \in R^{(n)}$  if  $\mathbf{a}$  is not a root of  $f$ , i.e., if  $f(\mathbf{a}) \in \mathcal{T}$ . We denote the set of these point as:*

$$\mathcal{T}_f = \{\mathbf{a} \in R^{(n)} : f(\mathbf{a}) \in \mathcal{T}\}.$$

*Confusion could arise because a tangible polynomial need not be tangible at every point. For example, the tangible polynomial  $(\lambda + 2)(\lambda + 1)$  is tangible at all tangible points except at 2 and 1, where its values are ghosts. It is easy to see that a polynomial in one indeterminate is tangible iff it is tangible at all but a finite number of the tangible points. Thus, any polynomial whose essential coefficients are all tangible is tangible.*

*Given a polynomial  $f(\lambda_1, \dots, \lambda_n) = \sum \alpha_i \lambda_1^{i_1} \dots \lambda_n^{i_n}$ , we define  $\hat{f}$  to be  $\sum \hat{\alpha}_i \lambda_1^{i_1} \dots \lambda_n^{i_n}$ , a tangible polynomial according to Definition 1.4 (although  $\hat{f}(\mathbf{a})$  need not be tangible for  $\mathbf{a} \in \mathcal{T}$ ). Note that  $\hat{f}(\mathbf{a})^\nu = f(\mathbf{a})^\nu$ .*

Recall that the **supertropical determinant**  $|A|$  of a matrix  $A = (a_{ij})$  is defined to be the permanent, i.e.,  $|A| = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)}$ ; cf. [4].

## 2. TRANSFORMATIONS OF SUPERTROPICAL VARIETIES

The **root set** of  $f \in R[\lambda_1, \dots, \lambda_n]$  is the set

$$Z(f) = \{\mathbf{a} \in R^{(n)} \mid f(\mathbf{a}) \in \mathcal{G}_0\},$$

and  $Z_{\text{tan}}(f) = Z(f) \cap \mathcal{T}_0^{(n)}$  is called the **tangible root set** of  $f$ .

The tangible root set provides a tropical version of affine geometry; analogously, one would define the supertropical version of projective geometry by considering equivalence classes of tangible roots of homogeneous polynomials (where, as usual, two roots are projectively equivalent if one is a scalar multiple

of the other). There is the usual way of viewing a polynomial  $f(\lambda_1, \dots, \lambda_n)$  of degree  $t$  as the homogeneous polynomial  $\lambda_{n+1}^t f(\frac{\lambda_1}{\lambda_{n+1}}, \dots, \frac{\lambda_n}{\lambda_{n+1}})$ , and visa versa. Since the algebra is easier to notate in the affine case, we focus on that.

We need to be able to find transformations of supertropical root sets, in order to move them away from “bad” points.

**Remark 2.1.** Suppose  $f(\lambda_1, \dots, \lambda_n) \in R[\lambda_1, \dots, \lambda_n]$ .

- (i) Given  $\mathbf{b} = (\beta_1, \dots, \beta_n) \in \mathcal{T}^{(n)}$ , we define the **multiplicative translation**

$$f_{(\mathbf{b}, \cdot)} = f(\beta_1 \lambda_1, \dots, \beta_n \lambda_n).$$

Clearly, when the  $\beta_i$  are invertible,

$$Z_{\tan}(f_{(\mathbf{b}, \cdot)}) = \{(\beta_1^{-1} a_1, \dots, \beta_n^{-1} a_n) : (a_1, \dots, a_n) \in Z_{\tan}(f)\}.$$

Thus, the roots of  $f_{(\mathbf{b}, \cdot)}$  are multiplicatively translated by  $\mathbf{b}$  from those of  $f$ .

- (ii) Given  $\beta \in R$ , define the **additive translation**

$$f_{(k, \beta, +)} = f(\lambda_1, \dots, \lambda_{k-1}, \lambda_k + \beta, \lambda_{k+1}, \dots, \lambda_n).$$

If  $\mathbf{a} \in Z_{\tan}(f)$  where  $\mathbf{a} = (a_1, \dots, a_n)$  with  $a_k^\nu$  “sufficiently small,” then  $\mathbf{a} \in Z_{\tan}(f_{(k, \beta, +)})$ . Indeed, writing  $f = \sum f_j \lambda_k^j$  where  $\lambda_k$  does not appear in  $f_j$ , and dividing through by the maximal possible power of  $\lambda_k$ , we may assume that  $f_0$  is nonzero, and thus dominates any root  $\mathbf{a} = (a_1, \dots, a_n)$  whose  $k$ -th component has small enough  $\nu$ -value. Hence  $f_0(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)$  is ghost, and this dominates in  $f_{(k, \beta, +)}$  as well as in  $f$ .

**2.1. The partial Frobenius morphism.** Another transformation comes from a morphism of supertropical root sets which arises from the **Frobenius property**, which we recall from [3, Remark 3.22]:

There is a semiring endomorphism

$$\phi : \text{Fun}(R^{(n)}, R) \longrightarrow \text{Fun}(R^{(n)}, R)$$

given by  $\phi : f \mapsto f^m$ . We want to refine this for polynomials.

**Definition 2.2.** Define the  $k$ -th  $m$ -Frobenius map  $\phi_k^m : R[\lambda_1, \dots, \lambda_n] \rightarrow R[\lambda_1, \dots, \lambda_n]$  given by

$$\phi_k^m : f(\lambda_1, \dots, \lambda_n) \longmapsto f(\lambda_1, \dots, \lambda_{k-1}, \lambda_k^m, \lambda_{k+1}, \dots, \lambda_n).$$

**Lemma 2.3.** For any  $k$  and  $m$ , the  $k$ -th  $m$ -Frobenius map  $\phi_k^m$  is a homomorphism of semirings, which is in fact an automorphism when  $m \in \mathbb{N}$  and  $R$  is a divisibly closed supertropical semifield.

*Proof.* Writing  $f = \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ , summed over  $\mathbf{i} = (i_1, \dots, i_n) \subset \mathbb{N}^{(n)}$ , we have

$$\phi_k^m(f) = \sum \alpha_{\mathbf{i}} \lambda_1^{i_1} \cdots \lambda_k^{mi_k} \cdots \lambda_n^{i_n}.$$

It follows just as in [3, Proposition 3.21] that for  $g = \sum \beta_{\mathbf{i}} \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ ,

$$\phi_k^m(f + g) = \phi_k^m(f) + \phi_k^m(g),$$

and clearly

$$\phi_k^m(fg) = \phi_k^m(f) \phi_k^m(g).$$

□

**Remark 2.4.** In the set-up of Lemma 2.3, each Frobenius map  $\phi_k^m$  defines a morphism of root sets, given by

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{k-1}, a_k^{\frac{1}{m}}, a_{k+1}, \dots, a_n),$$

which we call the **partial Frobenius morphisms**.

**2.2. Supertropical Zariski topology and the generic method.** Since one of the most basic tools in algebraic geometry is Zariski density, we would like to utilize the analogous tool here:

**Remark 2.5.** *Any polynomial formula expressing equality of  $\nu$ -values that holds on a dense subset of  $R^{(n)}$  must hold for all of  $R^{(n)}$ , since polynomials are continuous functions.*

Such a density argument is used in Section 4. There is an alternate method to Zariski density for verifying that identical relations holding for tangible polynomials must hold for arbitrary polynomials. It is not difficult to write down a generic polynomial over a semiring with tangibles and ghosts. Namely, we let  $\tilde{R} = R[\mu_0, \dots, \mu_t]$  where the  $\mu_i$  are indeterminates over  $R$ , and view the polynomial  $\sum_{i=0}^t \mu_i \lambda^i \in \tilde{R}[\lambda]$ ; any polynomial  $f = \sum \alpha_i \lambda^i \in R[\lambda]$  can be obtained by specializing the  $\mu_i$  accordingly. However, one has to contend with the following difficulty: Although this new semiring with ghosts  $\tilde{R}$  satisfies supertropicality, it is not a supertropical semiring, and so identical relations holding in supertropical semirings may well fail in  $\tilde{R}$ .

### 3. SUPERTROPICAL POLYNOMIALS IN ONE INDETERMINATE

This section is a direct continuation of [3]; we focus on properties of common tangible roots of polynomials in the supertropical setting. Assume throughout this section that  $F$  is an  $\mathbb{N}$ -divisible supertropical semifield, with ghost ideal  $\mathcal{G}_0$  and tangible elements  $\mathcal{T}$ . We view polynomials in  $F[\lambda]$  as functions, according to their equivalence classes in  $\text{CFun}(F, F)$ , or equivalently we consider the full polynomials [3, Definition 6.1] which are their natural representatives. Thus,

$$\begin{aligned} \text{a polynomial } f(\lambda) \text{ is ghost} &\iff f(a) \text{ is ghost for each } a \in F; \\ \text{a polynomial } f(\lambda) \text{ is tangible} &\iff f(W_{\mathcal{T}}) \text{ is not ghost for each tangible open interval } W_{\mathcal{T}} \subset F. \end{aligned}$$

A polynomial is called **monic** if its leading coefficient is  $1_F$  or  $1_F^\nu$  (i.e., 0 or  $0^\nu$  in logarithmic notation). We recall the following factorization:

**Theorem 3.1.** [3, Theorem 7.43 and Corollary 7.44] *Any monic full polynomial in one indeterminate has a unique factorization of the form  $f = f^{\text{tan}} f^{\text{intan}}$ , where the **tangible component**  $f^{\text{tan}}$  is the maximal product of tangible linear factors  $\lambda + a_i$ , and the **intangible component**  $f^{\text{intan}}$  is a product of irreducible quadratic factors of the form  $\lambda^2 + b_j^\nu \lambda + c_j$ , at most one linear left ghost  $\lambda^\nu + a_\ell$  and at most one linear right ghost  $\lambda + a_r^\nu$ . (One obtains  $f^{\text{tan}}$  and  $f^{\text{intan}}$  from the factorization called “minimal in ghosts.”)*

**Remark 3.2.** *The factors of  $f^{\text{intan}}$  as described in the theorem are all irreducible polynomials in  $F[\lambda]$ . Furthermore, by [3, Theorem 7.43], their sets of tangible roots are disjoint, and in fact one can read off these irreducible quadratic factors from the connected components of  $Z_{\text{tan}}(f)$ .*

Denoting the linear tangible terms as  $p_i = \lambda + a_i$  and the quadratic terms as  $q_j = \lambda^2 + b_j^\nu \lambda + c_j$ , we write

$$(3.1) \quad f = (\lambda^\nu + \alpha_\ell)(\lambda + \alpha_r^\nu) \prod_i p_i \prod_j q_j$$

for this factorization of  $f$  which is minimal in ghosts.

We say that a polynomial  $g(\lambda_1, \dots, \lambda_n)$  **e-divides**  $f(\lambda_1, \dots, \lambda_n)$  if, for a suitable polynomial  $h$ , the polynomials  $f$  and  $gh$  are e-equivalent. (A weaker concept is given below, in Definition 6.1).

**Remark 3.3.** (1) *Any tangible polynomial of degree  $n$  has at most  $n$  distinct tangible roots.*

(2) *If  $f \in F[\lambda]$  is a tangible polynomial of degree  $n$ , then  $f$  e-factors uniquely into  $n$  tangible linear factors.*

**Proposition 3.4.** *Suppose a polynomial  $p$  e-divides  $fg$  for  $g$  tangible, and  $p$  is irreducible nontangible. Then  $p$  e-divides  $f$ .*

*Proof.* Write  $f = f^{\text{tan}} f^{\text{intan}}$  as in (3.1) where  $f^{\text{tan}}$  is the tangible component of  $f$ . Then  $fg = f^{\text{tan}} f^{\text{intan}} g$  and  $f^{\text{tan}} g$  is tangible; hence  $f^{\text{tan}} g = (fg)^{\text{tan}}$  and  $p$  must divide  $(fg)^{\text{intan}} = f^{\text{intan}}$ .  $\square$

We turn to the question of how to compare polynomials in terms of their roots. The next example comes as a bit of a surprise.

**Example 3.5.** Some examples of polynomials  $f, g \in F[\lambda]$  such that  $f + g$  is ghost, but  $f$  and  $g$  have no common tangible root.

- (i) Suppose  $f$  is a full polynomial, all but one of whose monomials  $h$  have a ghost coefficient, and  $g = h$ . For example, take  $f = (\lambda^2)^\nu + 2\lambda + 3^\nu$  and  $g = 2\lambda$ . Then  $f + g$  is obviously ghost, but  $g$  has no tangible roots at all; thus,  $f$  and  $g$  have no common tangible roots.
- (ii) In logarithmic notation, where  $F = (\mathbb{R}, \max, +)$ , take  $f = \lambda(\lambda^\nu + 1) = (\lambda^2)^\nu + 1\lambda$ , and  $g = 1\lambda + 0^\nu$ . Then

$$(3.2) \quad f(a) = \begin{cases} 1a & \text{if } a^\nu < 1^\nu; \\ (a^2)^\nu & \text{if } a^\nu \geq 1^\nu. \end{cases}$$

In particular,  $f(a)$  is tangible for all  $a$  on the tangible open interval  $(-\infty, 1)$ . Also,

$$(3.3) \quad g(a) = \begin{cases} 0^\nu & \text{if } a^\nu \leq -1^\nu; \\ 1a & \text{if } a^\nu > -1^\nu. \end{cases}$$

In particular,  $g(a)$  is tangible for all  $a$  on the tangible open interval  $(-1, \infty)$ . Thus  $f$  and  $g$  have no common tangible roots, although  $f + g$  is ghost (since  $f(a) = g(a)$  for all  $a \in (-1, 1)$ ).

One can complicate this example, say by taking  $f = (\lambda^3)^\nu + 3(\lambda^2)^\nu + 3\lambda = (\lambda + 3)(\lambda^\nu + 0)\lambda$  and  $g = 3\lambda^2 + 2\lambda^\nu + 0^\nu$ . Nevertheless, these are the “only” kind of counterexamples, in the sense of the Proposition 3.8 below.

**3.1. Graphs and roots.** In this subsection, we assume that the supertropical semifield  $F$  is connected, in order to apply some topological arguments.

**Definition 3.6.** The **graph**  $\Gamma_f$  of a function  $f \in \text{CFun}(F^{(n)}, F)$  is defined as the set of ordered  $(n+1)$ -tuples  $(\mathbf{a}, f(\mathbf{a}))$  in  $F^{(n+1)}$ , where  $\mathbf{a} = (a_1, \dots, a_n) \in F^{(n)}$ . Note that either component of  $\Gamma_f$  could be tangible or ghost, so in a sense the graph has at most  $2^{n+1}$  leaves.

The  **$\mathcal{G}$ -graph**  $\Gamma_f^\nu$  is defined as  $\{(\mathbf{a}^\nu, f(\mathbf{a})^\nu) : \mathbf{a} \in F^{(n)}\}$ ; i.e., we project onto the ghost values. (Note that if  $\mathbf{a}^\nu = \mathbf{b}^\nu$ , then  $f(\mathbf{a})^\nu = f(\mathbf{b})^\nu$ .) It is more convenient to consider the **tangible  $\mathcal{G}$ -graph**

$$\Gamma_{f;T}^\nu = \{(\mathbf{a}^\nu, f(\mathbf{a})^\nu) : \mathbf{a} \in T^{(n)}\};$$

$\Gamma_{f;T}^\nu$  can be drawn in  $n+1$  dimensions.

In this paper, we consider a polynomial  $f \in F[\lambda]$  in one indeterminate, so its  $\mathcal{G}$ -graph lies on a plane, and is a sequence of line segments which can change slopes only at the tangible roots of  $f$ . We can describe the essential and quasi-essential monomials of  $f$  as in [3]: Writing  $f = \sum \alpha_i \lambda^i$ , and defining the slopes  $\gamma_i = \frac{\alpha_{i+1}}{\alpha_i}$ , we see that the monomial  $h = \alpha_i \lambda^i$  is essential only if  $\gamma_{i-1}^\nu < \gamma_i^\nu$ , and  $h$  is quasi-essential only if  $\gamma_{i-1}^\nu = \gamma_i^\nu$ . Note that when the monomial  $h$  is essential (at a point  $a$ ), the  $\mathcal{G}$ -graph  $\Gamma_f^\nu$  for  $f$  must change slope at  $a$ . We say that a polynomial is **full** if each of its monomials is essential or quasi-essential.

**Definition 3.7.** We say that  $f \in F[\lambda]$  is  **$\alpha$ -right** (resp. **left**) **half-tangible** for  $\alpha \in \mathcal{G}$  if  $f$  satisfies the following condition for each  $a \in T$ :

$$f(a) \in T \quad \text{iff} \quad a^\nu > \alpha \quad (\text{resp. } a^\nu < \alpha),$$

which implies  $f(a) \in \mathcal{G}_0$  for all  $a \in R$  with  $a^\nu < \alpha$  (resp.  $a^\nu > \alpha$ ).

By definition, if  $f$  is  $\alpha$ -right half-tangible, all roots of  $f$  must have  $\nu$ -value  $\leq \alpha$ , and thus the tangible  $\mathcal{G}$ -graph  $\Gamma_{f;T}^\nu$  of  $f$  must have a single ray emerging from  $\alpha$ . (The analogous assertion holds for left half-tangible.)

**Proposition 3.8.** If  $f, g \in F[\lambda]$  are polynomials without a common tangible root, with neither  $f$  nor  $g$  being monomials, and  $f + g$  is ghost, then  $f$  is left half-tangible and  $g$  is right half-tangible (or visa versa); explicitly, there are  $\alpha < \beta$  in  $\mathcal{G}$  such that  $f$  is  $\beta$ -left half-tangible,  $g$  is  $\alpha$ -right half-tangible, and  $f(a)^\nu = g(a)^\nu$  for all  $a$  in the tangible interval  $(\alpha, \beta)$ . Furthermore, in this case,  $\deg(f) > \deg(g)$  (and likewise the degree of the lowest order monomial of  $g$  is less than the degree of the lowest order monomial of  $f$ .)

*Proof.* In order for  $f + g$  to be ghost,  $(f + g)(a)$  must be ghost for each  $a \in F$ , which means that either:

- (1)  $f(a)$  is ghost of  $\nu$ -value greater than  $g(a)$ ,
- (2)  $g(a)$  is ghost of  $\nu$ -value greater than  $f(a)$ , or
- (3)  $f(a)^\nu = g(a)^\nu$ .

Let  $W_{f;\mathcal{T}}$  (resp.  $W_{g;\mathcal{T}}$ ) denote the (open) set of tangible elements satisfying Condition (1) (resp. (2)). We are done unless  $W_{f;\mathcal{T}}$  and  $W_{g;\mathcal{T}}$  are disjoint, since any element of the intersection would be a common tangible root of  $f$  and  $g$ .

Note that  $f(a)$  must be ghost for every element  $a$  in the closure of  $W_{f;\mathcal{T}}$ . (Indeed, if  $f(a)$  were tangible there would be some tangible interval  $U_{\mathcal{T}}$  containing  $a$  for which all values of  $f$  remain tangible; then,  $U_{\mathcal{T}} \cap W_{f;\mathcal{T}} \neq \emptyset$ , contrary to definition of  $W_{f;\mathcal{T}}$ .) Likewise,  $g(a)$  is ghost for every element  $a$  in the closure of  $W_{g;\mathcal{T}}$ .

If  $W_{f;\mathcal{T}} = \emptyset$ , then  $Z_{\text{tan}}(g) = \mathcal{T}$ , and any tangible root of  $f$  is automatically a root of  $g$ . Hence, we may assume that  $W_{f;\mathcal{T}}$  and likewise  $W_{g;\mathcal{T}}$  are nonempty.

Also, let

$$S_{\mathcal{T}} = \{a \in \mathcal{T} : f(a)^\nu = g(a)^\nu\}.$$

Let  $S_{f;\mathcal{T}} = \{a \in S_{\mathcal{T}} : f(a) \text{ is tangible}\}$  and  $S_{g;\mathcal{T}} = \{a \in S_{\mathcal{T}} : g(a) \text{ is tangible}\}$ . Since any  $a \in \mathcal{T}$  cannot be a common root of  $f$  and  $g$ , we must have  $f(a)$  or  $g(a)$  tangible, thereby implying  $S_{f;\mathcal{T}} \cup S_{g;\mathcal{T}} = S_{\mathcal{T}}$ . As noted above,  $S_{f;\mathcal{T}}$  is disjoint from the closure of  $W_{f;\mathcal{T}}$ .

Suppose  $a$  is a tangible element in the boundary of  $W_{f;\mathcal{T}}$  (which by definition is the complement of  $W_{f;\mathcal{T}}$  in its closure). Then  $f(a)^\nu = g(a)^\nu$ . As noted above,  $f(a)$  must be ghost; if  $a$  also lies in the closure of  $W_{g;\mathcal{T}}$ , then  $g(a)$  is also ghost, contrary to the hypothesis that  $f$  and  $g$  have no common tangible roots. Since  $\mathcal{T}$  is presumed connected, we must have  $S_{f;\mathcal{T}} \cap S_{g;\mathcal{T}} \neq \emptyset$ .

Write  $S_{f;\mathcal{T}} \cap S_{g;\mathcal{T}}$  as a union of disjoint intervals, one of which we denote as  $(\alpha, \beta)$ . For  $a'$  of  $\nu$ -value slightly more than  $\beta$ , suppose  $a' \in W_{f;\mathcal{T}}$ . Then the slope of the tangible  $\mathcal{G}$ -graph  $\Gamma_{f;\mathcal{T}}^\nu$  of  $f$  at  $a'$  must be at least as large as the slope of  $\Gamma_{g;\mathcal{T}}^\nu$  at  $a'$ , and this situation continues unless  $g$  has some tangible root  $a \in W_{f;\mathcal{T}}$ , contrary to hypothesis. Thus,  $g(a)^\nu < f(a)^\nu$  for each  $a$  of  $\nu$ -value  $> \beta$ , implying  $f(a) \in \mathcal{G}_0$  for all such  $a$ , and thus, by hypothesis,  $g(a) \in \mathcal{T}$  for all  $a$  of  $\nu$ -value  $> \beta$ .

We have also proved that  $S_{f;\mathcal{T}} \cap S_{g;\mathcal{T}} = (\alpha, \beta)$  is connected, and its closure is all of  $S_{\mathcal{T}}$  since otherwise  $S_{\mathcal{T}}$  has a tangible point at which the  $\mathcal{G}$ -graphs,  $\Gamma_{f;\mathcal{T}}^\nu$  and  $\Gamma_{g;\mathcal{T}}^\nu$ , both change slopes and thus must both have a tangible root. Hence,  $f$  and  $g$  are both tangible on the interior of  $S_{\mathcal{T}}$ .

By hypothesis,  $g$  is not a monomial, and thus has some tangible root, which must have  $\nu$ -value  $< \alpha$ . The previous argument applied in the other direction (for small  $\nu$ -values) shows that  $g(a)$  is ghost and  $f(a)$  is tangible for all  $a$  of  $\nu$ -value  $< \alpha$ .

Finally, since  $f$  increases faster than  $g$  for  $a^\nu > \beta$ , it follows at once that  $\deg(f) > \deg(g)$ ; the last assertion follows by symmetry.  $\square$

Conversely, if  $f, g$  satisfy the conclusion of Proposition 3.8, then clearly  $f + g$  are ghost. Thus, a pair of polynomials whose sum is ghost is characterized either as having a common tangible root or else satisfying the conclusion of Proposition 3.8. In particular, two polynomials of the same degree whose sum is ghost must have a common tangible root.

**Example 3.9.** *It is also instructive to consider the following example:*

$$f = (\lambda + 2)(\lambda + 5^\nu)(\lambda + 8^\nu)(\lambda + 9), \quad \text{and} \quad g = (\lambda + 3)(\lambda + 4)(\lambda^\nu + 7)(\lambda + 10).$$

*We have the following table of values for  $f$  and  $g$ :*

$a$	2	3	4	5	6	7	8	9	10	11	...
$f(a)$	$24^\nu$	$25^\nu$	$26^\nu$	$27^\nu$	$29^\nu$	$31^\nu$	$33^\nu$	$36^\nu$	40	44	...
$g(a)$	24	$24^\nu$	$25^\nu$	27	29	$31^\nu$	$34^\nu$	$37^\nu$	$40^\nu$	$44^\nu$	...

*Note that  $\deg(f) = \deg(g) = 4$  and  $f + g = \nu(\lambda^4 + 10\lambda^3 + 17\lambda^2 + 22\lambda + 24)$  is ghost, whereas they have exactly three ordinary common tangible roots, namely 3, 4, and 9; each  $a \in [7, 8]$  is also a common tangible root.*

**3.2. Relatively prime polynomials.** In order to compare polynomials in terms of their roots, we need another notion. Given a polynomial  $f$ , we write  $\deg(f)$  for the degree of the lowest order monomial of  $f$ . For example,  $\deg(\lambda^3 + 2\lambda^2 + \lambda^\nu) = 1$ .

**Definition 3.10.** Two polynomials  $f$  and  $g$  of respective degrees  $m$  and  $n$  are **relatively prime** if there do not exist tangible polynomials  $\hat{p}$  and  $\hat{q}$  (not both  $0_F$ ) with  $\deg(\hat{p}) < n$  and  $\deg(\hat{q}) < m$ , such that  $\hat{p}f + \hat{q}g$  is ghost with  $\deg(\hat{p}f) = \deg(\hat{q}g)$  and  $\deg(\hat{p}f) = \deg(\hat{q}g)$ .

We say that two polynomials  $f$  and  $g$  have a **common  $\nu$ -factor**  $h$  if there are polynomials  $h_1, h_2$  with  $h_1^\nu = h_2^\nu = h^\nu$ , such that  $h_1$   $\epsilon$ -divides  $f$  and  $h_2$   $\epsilon$ -divides  $g$ .

**Remark 3.11.** Any monic polynomials  $f$  and  $g$  having a common  $\nu$ -factor  $h$  are not relatively prime. Indeed, write  $f = h_1q$  and  $g = h_2p$  and thus

$$\hat{p}f + \hat{q}g = \hat{p}h_1q + \hat{q}h_2p = h_1\hat{p}q + h_2p\hat{q} = \text{ghost}$$

(since they are  $\nu$ -matched). On the other hand, two non-relatively prime polynomials without a common factor could be irreducible; for example for  $\lambda + 2^\nu$  and  $\lambda + 1$  we have  $(\lambda + 2^\nu)1 + (\lambda + 1)1$  is ghost, but both are irreducible, cf. Remark 3.2.

**Remark 3.12.**

- (1) If  $\deg(f)$  and  $\deg(g)$  are both positive, then  $f$  and  $g$  cannot be relatively prime, since they have the common  $\nu$ -factor  $\lambda$ . Similarly, if  $\deg(f) = 0$  and  $\deg(g) > 0$ , then one can cancel  $\lambda$  from  $g$  without affecting whether  $g$  is relatively prime to  $f$ . Thus, the issue of being relatively prime can be reduced to polynomials having nontrivial constant term. But then, cancelling powers of  $\lambda$  from  $\hat{p}$  and  $\hat{q}$ , we may assume that  $\hat{p}$  and  $\hat{q}$  also have nontrivial constant term. Thus,  $\deg(\hat{p}f) = \deg(\hat{q}g) = 0$ , so the condition that their lower degrees match is automatic.
- (2) Adjusting the leading coefficients in the definition, we may assume that  $f$  and  $g$  are both monic. (However,  $\hat{p}$  and  $\hat{q}$  need not be monic, as evidenced taking  $f = \lambda^\nu + 1$  and  $g = \lambda + 3$  in logarithmic notation; then  $2f + g$  is ghost.)
- (3) A nonconstant ghost polynomial  $f$  cannot be relatively prime to any nonconstant polynomial  $g$ , since  $fh + g$  or  $f + hg$  is ghost, where  $h$  is any polynomial of degree  $|\deg f - \deg g|$  with “large enough” coefficients or “small enough” coefficients respectively.
- (4) If  $\hat{p}f + \hat{q}g$  is ghost with  $\hat{p} = (\lambda + a)\hat{p}_1$  and  $\hat{q} = (\lambda + a)\hat{q}_1$  then  $(\lambda + a)(\hat{p}_1f + \hat{q}_1g)$  is ghost. Hence,  $\hat{p}_1f + \hat{q}_1g$  is ghost at every point except  $a$ , which implies  $\hat{p}_1f + \hat{q}_1g$  is ghost, by continuity.

We also need the following observation to ease our computations.

**Lemma 3.13.** Suppose the polynomial  $f + g$  is ghost, and  $p, q \in R[\lambda]$  with  $p^\nu = q^\nu$ . Then  $pf + qg$  is also ghost.

*Proof.* Write  $p = \sum \alpha_i \lambda^i$  and  $q = \sum \beta_i \lambda^i$  where  $\alpha_i^\nu = \beta_i^\nu$ . For any monomial  $f_\ell$  of  $f$  there is some monomial  $g_k$  of  $g$  such that  $f_\ell + g_k$  is ghost. But any monomial of  $pf$  has the form  $\alpha_i \lambda^i f_\ell$ , which when added to  $\beta_i \lambda^i g_k$  is clearly ghost.  $\square$

**Theorem 3.14.** Over a connected supertropical semifield  $F$ , two non-constant monic polynomials  $f$  and  $g$  in  $F[\lambda]$  are not relatively prime iff  $f$  and  $g$  have a common tangible root.

*Proof.* We may assume that  $f$  and  $g$  are both monic. In view of Remark 3.12, we may also assume that  $f$  and  $g$  are non-ghost, and have nontrivial constant term.

( $\Rightarrow$ ) Suppose  $f$  and  $g$  are not relatively prime; i.e.,  $\hat{p}f + \hat{q}g$  is ghost for some tangible polynomials  $\hat{p}$  and  $\hat{q}$ , with  $\deg(\hat{p}f) = \deg(\hat{q}g)$  and  $\deg(\hat{p}f) = \deg(\hat{q}g)$ . Since  $\deg(f) = \deg(g) = 0$ , we may cancel out the same power of  $\lambda$  from both  $\hat{p}$  and  $\hat{q}$ , and thereby assume that  $\hat{p}f$  and  $\hat{q}g$  each have nontrivial constant term. We proceed as in the proof of Proposition 3.8, but with more specific attention to the tangible  $\mathcal{G}$ -graphs  $\Gamma_{\hat{p}f; \mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g; \mathcal{T}}^\nu$ , cf. Definition 3.6. We assume that  $f$  and  $g$  have no common tangible root. In other words,  $f(a) \in \mathcal{G}$  implies  $g(a) \in \mathcal{T}$ , for any  $a \in \mathcal{T}$ , and likewise  $g(a) \in \mathcal{G}$  implies  $f(a) \in \mathcal{T}$ . Also, we may assume that  $\hat{p}$  and  $\hat{q}$  have no common tangible root, by Remark 3.12(4).



Let  $W_{\hat{p}f;\mathcal{T}} = \{a \in \mathcal{T} : \hat{p}f(a)^\nu > \hat{q}g(a)^\nu\}$ , and  $W_{\hat{q}g;\mathcal{T}} = \{a \in \mathcal{T} : \hat{q}g(a)^\nu > \hat{p}f(a)^\nu\}$ . By hypothesis,  $\hat{p}f(a')$  is ghost for all  $a' \in W_{\hat{p}f;\mathcal{T}}$ . But any  $a' \in W_{\hat{p}f;\mathcal{T}}$  is contained in a tangible open interval  $U_{\mathcal{T}}$  for which  $\hat{p}$  is tangible on  $U_{\mathcal{T}} \setminus \{a'\}$ , so by assumption,  $f(a) \in \mathcal{G}$  for all  $a \in U_{\mathcal{T}} \setminus \{a'\}$ , and thus  $f(a) \in \mathcal{G}$  for all  $a \in U_{\mathcal{T}}$ . For all  $a \in W_{\hat{p}f;\mathcal{T}}$ , it follows that  $f(a) \in \mathcal{G}$  and thus  $g(a) \in \mathcal{T}$ . Likewise, for all  $b \in W_{\hat{q}g;\mathcal{T}}$ , we have  $g(b) \in \mathcal{G}$  and  $f(b) \in \mathcal{T}$ .

Note that as we increase the  $\nu$ -value of a point, the slope of the graph  $\Gamma_{f;\mathcal{T}}^\nu$  of a polynomial  $f$  can only increase; moreover, an increase of slope in the graph indicates the corresponding increase of degree of the dominant monomial at that point. We write  $\text{dom}_{\hat{p}f}(a)$  (resp.  $\text{dom}_{\hat{q}g}(a)$ ) for the maximal degree of a dominant monomial of  $\hat{p}f$  (resp.  $\hat{q}g$ ) at  $a \in F$ .

Let

$$S_{\mathcal{T}} = \mathcal{T} \setminus (W_{\hat{p}f;\mathcal{T}} \cup W_{\hat{q}g;\mathcal{T}}) = \{a \in \mathcal{T} : \hat{p}f(a)^\nu = \hat{q}g(a)^\nu\}.$$

Clearly,  $\text{dom}_{\hat{p}f}(a) = \text{dom}_{\hat{q}g}(a)$  for every  $a$  in the interior of  $S_{\mathcal{T}}$ , since the graphs  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  must have the same slope there.

By symmetry, we may assume that  $f(a_{\text{sml}}) \in \mathcal{G}$  for  $a_{\text{sml}}^\nu$  small. The objective of our proof is to show that as  $a \in F$  increases, any change in the slope of  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  arising from an increase of degree of the essential monomial of  $f$  is matched by corresponding roots of  $\hat{q}$ , and thus  $\deg(\hat{q}) = \deg(f)$  (and  $\deg(\hat{p}) = \deg(g)$ ) — a contradiction.

We claim that the graphs  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  do not cross at any single tangible point (i.e. without some interval in  $S_{\mathcal{T}}$ ). Indeed, consider an arbitrary tangible point  $a \in S_{\mathcal{T}}$  at which the graphs of  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  would cross, starting say with  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  above  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  before  $a$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  above  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  after  $a$ . At this intersection point,  $\hat{p}f(a)$  and  $\hat{q}g(a)$  must both be ghost, so  $f(b)$  must be ghost for  $b$  of  $\nu$ -value  $< a^\nu$  whereas  $g(b)$  must be ghost for  $b$  of  $\nu$ -value  $> a^\nu$ . But this yields a common root for  $f$  and  $g$  unless  $f$  switches from ghost to tangible and  $g$  switches from tangible to ghost, so  $a$  would be a common root of  $f$  and  $g$ , yielding a contradiction.

This proves that any point  $a$  at which the graphs  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  meet must lie on the boundary of  $S_{\mathcal{T}}$ . Continuing along  $S_{\mathcal{T}}$ , suppose that  $f$  has a root  $b$  in the interior of  $S_{\mathcal{T}}$ . Then the slope of  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  increases by some number matching the increase  $k$  of degree in the essential monomial of  $f$  at  $b$ ; this must be matched by an equal increase in slope in  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$ . But  $g$  cannot have a root here, since  $f$  and  $g$  have no common tangible roots; hence  $b$  is a root of  $\hat{q}$  of multiplicity  $k$ . Thus, all roots of  $f$  in the interior of  $S_{\mathcal{T}}$  are matched by roots of  $\hat{q}$ .

Next let us consider what happens between two points on subsequent tangible intervals of  $S_{\mathcal{T}}$ . At any boundary point  $a'$  of  $S_{\mathcal{T}}$ , for  $a$  of slightly greater  $\nu$ -value than  $a'$ , we have  $a \in W_{\hat{p}f;\mathcal{T}} \cup W_{\hat{q}g;\mathcal{T}}$ ; say  $a \in W_{\hat{p}f;\mathcal{T}}$ . This means  $\text{dom}_{\hat{p}f}(a') > \text{dom}_{\hat{q}g}(a')$ . Clearly  $a'$  is a root of  $\hat{p}f$ , and furthermore, since  $g$  is tangible in  $W_{\hat{p}f;\mathcal{T}}$ , any increase in  $\text{dom}_{\hat{q}g}(a')$  occurs because of changes in the essential monomial of  $\hat{q}$ , i.e., from roots of  $\hat{q}$ . Thus, when we enter  $S_{\mathcal{T}}$  the next time, say at  $a''$ , we see that  $\text{dom}_{\hat{p}f}(a'') - \text{dom}_{\hat{p}f}(a')$  is the number of roots of  $\hat{q}$  needed to increase the slope of  $\hat{q}g$  accordingly. But when we are within  $W_{\hat{q}g;\mathcal{T}}$ , there cannot be any tangible roots of  $f$ , and thus the essential monomial of  $f$  does not change. Continuing until we reach  $a''$ , we see that the only increase in degree coming from change of the dominant monomial of  $f$  must occur in  $W_{\hat{p}f;\mathcal{T}} \cup S_{\mathcal{T}}$  and are thus matched by roots from  $\hat{q}$ .

Looking at the whole picture, we see that both graphs  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  and  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  have slope 0 for small  $\nu$ -values of  $a$  (since both  $\hat{p}f$  and  $\hat{q}g$  have nontrivial constant terms). Either they coincide for small  $\nu$ -values of  $a$ , and we start in  $S_{\mathcal{T}}$ , or else one is above the other. Assume that  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  starts above  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$ . But any increase of slope of  $\Gamma_{\hat{p}f;\mathcal{T}}^\nu$  entails the same increase of slope of  $\Gamma_{\hat{q}g;\mathcal{T}}^\nu$  (since otherwise we would have a crossing at a single tangible point), and thus a corresponding increase in  $\deg(\hat{p})$ , since any tangible root of  $f$  (before the crossing) would be a common root of  $f$  and  $g$ , contrary to hypothesis. Then the crossing brings us to  $S_{\mathcal{T}}$ , and we continue the argument until the last interval in  $S_{\mathcal{T}}$ , and then when we leave, the analogous argument at the end shows that any increase in the upper graph leads to a corresponding increase in the tangible polynomial ( $\hat{p}$  or  $\hat{q}$ ) in the other graph.

Combining these different stages shows that  $\deg(\hat{q}) \leq \deg(f)$ , which is what we were trying to prove.

(Symmetrically, any contribution to  $\text{dom}_{\hat{q}g}$  coming from changes in the essential monomial of  $g$  happens in  $W_{\hat{q}g} \cup S_{\mathcal{T}}$ , and thus is matched by roots of  $\hat{p}$ .)

( $\Leftarrow$ ) Our strategy is to e-factor  $f$  and  $g$  into  $e$ -irreducible polynomials, all of which have degree  $\leq 2$ . Thus, we suppose first that  $f$  and  $g$  are  $e$ -irreducible polynomials of respective degrees  $m$  and  $n$  ( $\leq 2$ ) having a common tangible root  $a$ , and consider the following cases according to Theorem 3.1:

*Case I:* Suppose  $m = n = 1$ . If  $f$  and  $g$  are both tangible we are done, since then  $f = g = \lambda + a$ . The cases when both  $f$  and  $g$  are linear left ghost or linear right ghost are also clear. Finally, when  $f = \lambda^\nu + \alpha_f$  and  $g = \lambda + \alpha_g^\nu$ , for  $\alpha_f, \alpha_g \in \mathcal{T}$ , we must have  $\alpha_f^\nu \leq a^\nu \leq \alpha_g^\nu$ . Thus  $f + g = \lambda^\nu + \alpha_g^\nu$ .

*Case II:* Suppose  $m = 2$  and  $n = 1$ , and let  $f = \lambda^2 + \beta_f^\nu \lambda + \alpha_f$  with  $(\beta_f^2)^\nu > \alpha_f^\nu$ . For  $g = \lambda + \alpha_g^\nu$ , we have  $(\frac{\alpha_f}{\beta_f})^\nu \leq a^\nu \leq \min\{\beta_f^\nu, \alpha_g^\nu\}$ , so  $\alpha_f^\nu \leq (\beta_f \alpha_g)^\nu$  and

$$f + (\lambda + \beta_f)g = (\lambda^2)^\nu + (\beta_f^\nu + \beta_f + \alpha_g^\nu)\lambda + \alpha_f + (\beta_f \alpha_g)^\nu = (\lambda^2)^\nu + (\beta_f^\nu + \alpha_g^\nu)\lambda + (\alpha_g \beta_f)^\nu$$

is ghost.

When  $g = \lambda^\nu + \alpha_g$ , then  $\max\left\{\alpha_g^\nu, \left(\frac{\alpha_f}{\beta_f}\right)^\nu\right\} \leq a^\nu \leq \beta_f^\nu$ , so

$$f + \left(\lambda + \frac{\alpha_f}{\alpha_g}\right)g = (\lambda^2)^\nu + \left(\beta_f^\nu + \frac{\alpha_f^\nu}{\alpha_g}\right)\lambda + \alpha_f^\nu - \text{a ghost}.$$

If  $g = \lambda + \alpha_g$ , then  $a = \alpha_g$  and  $\left(\frac{\alpha_f}{\beta_f}\right)^\nu \leq a^\nu \leq \beta_f^\nu$ , implying  $f + (\lambda + \frac{\alpha_f}{\alpha_g})g = (\lambda^2)^\nu + \beta_f^\nu \lambda + \alpha_f^\nu$  is ghost.

*Case III:* Suppose  $m = n = 2$ , and let  $f = \lambda^2 + \beta_f^\nu \lambda + \alpha_f$  and  $g = \lambda^2 + \beta_g^\nu \lambda + \alpha_g$ , for  $\alpha_f, \alpha_g \in \mathcal{T}$  with  $(\beta_f^2)^\nu > \alpha_f^\nu$  and  $(\beta_g^2)^\nu > \alpha_g^\nu$ . Then

$$(3.4) \quad \max\left\{\left(\frac{\alpha_f}{\beta_f}\right)^\nu, \left(\frac{\alpha_g}{\beta_g}\right)^\nu\right\} \leq a^\nu \leq \min\{\beta_f^\nu, \beta_g^\nu\}.$$

By symmetry, we may assume that  $\alpha_f^\nu \geq \alpha_g^\nu$ . We claim that there are elements  $x, y \in \mathcal{T}_0$  such that, for  $\hat{p} = \lambda + x$  and  $\hat{q} = \lambda + y$ , the polynomial

$$(3.5) \quad \hat{p}f + \hat{q}g = (\lambda^3)^\nu + (\beta_f^\nu + x + \beta_g^\nu + y)\lambda^2 + (x\beta_f^\nu + \alpha_f + y\beta_g^\nu + \alpha_g)\lambda + (x\alpha_f + y\alpha_g)$$

is ghost.

Indeed, take  $y = \hat{\nu}(\max\{\beta_f^\nu, \beta_g^\nu\})$  and  $x = \frac{\alpha_g}{\alpha_f}y$ . The constant term in (3.5) is  $(y\alpha_g + y\alpha_g) = y\alpha_g^\nu$ . Likewise, the coefficient of  $\lambda^2$  is ghost since  $\beta_f^\nu + \beta_g^\nu$  dominates  $y$  and  $x$ . Finally, the linear term is ghost since  $\beta_g^\nu \geq \left(\frac{\alpha_f}{\beta_f}\right)^\nu$  by the inequality (3.4), implying

$$y\beta_g^\nu \geq \beta_f^\nu \frac{\alpha_f^\nu}{\beta_f^\nu} = \alpha_f^\nu.$$

(The case for  $f$  or  $g$  ghost is trivial, by Remark 3.12.)

In general, suppose  $f$  and  $g$  are not necessarily irreducible, and have the common tangible root  $a \in \mathcal{T}$ . Consider the factorizations of  $f = \prod_i f_i$  and  $g = \prod_j g_j$  into irreducible (linear and quadratic) polynomials. Thus,  $a$  is a common tangible root of some  $f_i$  and  $g_j$  of respective degrees  $m_i, n_j \leq 2$ , and, by the first part of the proof,  $\hat{p}_i f_i + \hat{q}_j g_j$  is ghost for suitable tangible polynomials  $\hat{p}_i$  and  $\hat{q}_j$  with  $\deg(\hat{p}_i) < n_i$  and  $\deg(\hat{q}_j) < m_j$  and  $\deg(\hat{p}_i f_i) = \deg(\hat{q}_j g_j)$  and  $\deg(\hat{p}_i f_i) = \deg(\hat{q}_j g_j)$ . Let

$$r = \prod_{t \neq i} f_t; \quad s = \prod_{u \neq j} g_u.$$

Taking  $\hat{p} = \hat{p}_i \hat{s}$  and  $\hat{q} = \hat{q}_j \hat{r}$ , we have  $\hat{p}f + \hat{q}g$  ghost. Indeed, since  $\hat{p}_i f_i + \hat{q}_j g_j$  is ghost, and  $\hat{r}s$  and  $\hat{r}s$  have the same  $\nu$ -value, write

$$\hat{p}f + \hat{q}g = \hat{p}_i \hat{s} f_i \prod_{t \neq i} f_t + \hat{q}_j \hat{r} g_j \prod_{u \neq j} g_u = \hat{p}_i f_i \hat{s} r + \hat{q}_j g_j \hat{r} s$$

which is ghost by Lemma 3.13, and the degrees clearly match.  $\square$

The contrapositive of Theorem 3.14 gives us the following analog of part of Bézout's theorem:

**Corollary 3.15.** *Over a connected supertropical semifield  $F$ , if  $f$  and  $g$  are two polynomials with no tangible roots in common, then they are relatively prime.*

One should compare this result with [3, Theorem 8.5], which says that for any polynomials  $f$  and  $g$  in  $F[\lambda]$  with no tangible roots in common, there exist tangible polynomials  $\hat{p}$  and  $\hat{q}$  such that, in logarithmic notation,

$$(3.6) \quad \hat{p}f + \hat{q}g = 0 + \lambda \cdot \text{ghost}.$$

But this last result is not as sharp as Corollary 3.15, since  $f = \lambda + 0$  and  $g = 2^\nu \lambda$  have the tangible root 0 in common, whereas they satisfy (3.6), taking  $\hat{p} = \hat{q} = 0$ .

**Example 3.16.** Consider the polynomials  $f = 2^\nu \lambda^2 + 4\lambda$  and  $g = \lambda + 1^\nu$ , and  $\hat{q} = \lambda + 4$ . Then  $g\hat{q} = \lambda^2 + 4\lambda + 5^\nu$ , so  $f + g\hat{q} = 2^\nu \lambda^2 + 4^\nu \lambda + 5^\nu$  is ghost. On the other hand,  $f = 2\lambda(\lambda^\nu + 2)$  has the same roots as  $\lambda^\nu + 2$ , whose tangible roots (the interval  $[2, \infty)$ ) are disjoint from those of  $g$  (the interval  $(-\infty, 1]$ ). Note that  $\deg(f) = \deg(\hat{q}g)$ , but  $1 = \underline{\deg}(f) > \underline{\deg}(\hat{q}g)$ .

#### 4. THE RESULTANT OF TWO POLYNOMIALS

In this section, we consider polynomials in one indeterminate over an arbitrary supertropical semiring  $R$ , with an eye towards applying induction and eventually dealing with polynomials in an arbitrary number of indeterminates. Our main task is to determine when two polynomials are relatively prime. This depends on the essential parts of the polynomials. Since we want to use full polynomials in the sense of [3, Definition 7.10], we assume throughout this section that the polynomials  $f$  and  $g$  in  $R[\lambda]$ , of respective degrees  $m$  and  $n$ , are quasi-essential, in the sense that every monomial is quasi-essential. In other words, writing  $f = \sum \alpha_i \lambda^i$ , we may assume that

$$\left( \frac{\hat{\alpha}_i}{\hat{\alpha}_{i+1}} \right)^\nu \leq \left( \frac{\hat{\alpha}_{i+1}}{\hat{\alpha}_{i+2}} \right)^\nu$$

for each  $i$ . The classical method for checking relative primeness of polynomials is via the resultant, which has a natural supertropical version.

**Remark 4.1.** For any semiring  $R$ , suppose  $f = \sum_{i=0}^m \alpha_i \lambda^i \in R[\lambda]$ , and let  $A_n(f)$  denote the  $n \times (m+n)$  matrix

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_m & & & & \dots \\ & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{m-1} & \alpha_m & & & \\ & & \alpha_0 & \alpha_1 & \dots & \alpha_{m-2} & \alpha_{m-1} & \alpha_m & & \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & & \alpha_0 & \dots & \dots & \dots & \alpha_m & \\ \dots & & & & & \alpha_0 & \alpha_1 & \dots & \dots & \alpha_{m-1} & \alpha_m \end{pmatrix},$$

where the empty places are filled by  $0_R$ . Then for any polynomial  $p = \sum_{i=0}^{n-1} \gamma_i \lambda^i$ , we have

$$(\gamma_0 \ \gamma_1 \ \dots \ \gamma_{n-1}) A_n(f) = (\mu_0 \ \mu_1 \ \dots \ \mu_{m+n-1}),$$

where  $pf = \sum_{i=0}^{m+n-1} \mu_i \lambda^i$ . (This is seen by inspection, just as in the classical ring-theoretic case, since negatives are not used in the proof).

**Definition 4.2.** The **resultant matrix**  $\mathfrak{R}(f, g)$  is the  $m+n$  square matrix

$$\mathfrak{R}(f, g) = \begin{pmatrix} A_n(f) \\ A_m(g) \end{pmatrix}.$$

The **resultant** of  $f$  and  $g$  is the supertropical determinant  $|\mathfrak{R}(f, g)|$ . When  $g = \beta$  is constant, we formally define  $|\mathfrak{R}(f, g)| = \beta^m$ . (Thus, when both  $f$  and  $g$  are constant,  $|\mathfrak{R}(f, g)| = 1_R$ .)

**Remark 4.3.**

- (i)  $|\mathfrak{R}(f, g)| = |\mathfrak{R}(g, f)|$ , since we pass from  $\mathfrak{R}(f, g)$  to  $\mathfrak{R}(g, f)$  by permuting rows.

(ii) If  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \sum_{j=0}^n \beta_j \lambda^j$ , then

$$|\mathfrak{R}(f, g)| = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_m & & \dots \\ & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_m & \\ \vdots & & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \\ & & & \ddots & \dots & \ddots & \vdots \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n & & \\ & \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n & \\ & & \beta_0 & \beta_1 & \beta_2 & \dots & \vdots \\ & & & \ddots & \dots & & \end{vmatrix}.$$

For any tangible  $c$ , dividing each of the last  $m$  rows by  $c$  shows  $|\mathfrak{R}(f, g)| = c^m |\mathfrak{R}(f, \frac{1}{c}g)|$ . Thus, it is easy to reduce to the case that  $g$  is monic, and likewise for  $f$ . We often make this assumption without further ado.

We need to compute the precise  $\nu$ -value of  $|\mathfrak{R}(f, g)|$ . Towards this end, the following remark is useful.

**Remark 4.4.**  $|\mathfrak{R}(f, g)|^\nu = |\mathfrak{R}(\hat{f}, \hat{g})|^\nu$ . Indeed, by definition, the entries of the matrices whose determinants define  $|\mathfrak{R}(f, g)|$  and  $|\mathfrak{R}(\hat{f}, \hat{g})|$  have the same  $\nu$ -values, so their determinants have the same  $\nu$ -values.

**Remark 4.5.** By Remark 4.1, for any  $p = \sum_{i=0}^{n-1} \alpha_i \lambda^i$  and  $q = \sum_{i=0}^{m-1} \beta_i \lambda^i$  in  $R[\lambda]$  of respective degrees  $n-1$  and  $m-1$ , with  $pf + qg = \sum_{i=0}^{m+n-1} \mu_i \lambda^i$ , we have

$$(4.1) \quad (\alpha_0 \ \dots \ \alpha_{n-1} \ \beta_0 \ \dots \ \beta_{m-1}) \mathfrak{R}(f, g) = (\mu_0 \ \mu_1 \ \dots \ \mu_{m+n-1}).$$

The direction of our inquiry is indicated by the next observation.

**Remark 4.6.** If  $f, g \in F[\lambda]$  are not relatively prime, then  $|\mathfrak{R}(f, g)|$  is a ghost. (Just take tangible  $p, q$  of respective degrees  $\leq n-1$  and  $m-1$  such that  $pf + qg$  is a ghost, and apply Remark 4.5.)

We look for the converse: That is, if  $|\mathfrak{R}(f, g)|$  is ghost, then  $f$  and  $g$  are not relatively prime, and thus have a common tangible root.

**Example 4.7.**

(i) Suppose  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \beta_1 \lambda + \beta_0$ . The resultant  $|\mathfrak{R}(f, g)|$  is given by:

$$|\mathfrak{R}(f, g)| = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_m \\ \beta_0 & \beta_1 & & & \\ & \beta_0 & \beta_1 & & \\ & & \ddots & \ddots & \\ & & & \beta_0 & \beta_1 \end{vmatrix} = \alpha_0 \beta_1^m + \alpha_1 \beta_0 \beta_1^{m-1} + \alpha_2 \beta_0^2 \beta_1^{m-2} + \dots + \alpha_m \beta_0^m.$$

In particular, if  $\beta_1 = 1$ , then  $|\mathfrak{R}(f, g)| = f(\beta_0)$ , which is a ghost iff  $\beta_0$  is a root of  $f$  (as well as of  $g$ ). We conclude for  $\beta_0, \beta_1$  tangible that  $|\mathfrak{R}(f, g)|$  is a ghost iff  $f$  and  $g$  have a common root. (Indeed, first divide through by  $\beta_1$  to reduce to the case  $\beta_1 = 1$ , and then apply the previous sentence.)

(ii) Suppose  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \lambda + b^\nu$ , for  $b$  tangible. As in (i), the resultant  $|\mathfrak{R}(f, g)|$  equals  $f(b^\nu)$ , which is a ghost iff  $b^\nu \geq (\frac{\alpha_0}{\alpha_1})^\nu$ , the root of  $f$  having smallest  $\nu$ -value. Again, the resultant is a ghost iff  $f$  and  $g$  have a common tangible root.

(iii) Suppose  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \lambda^\nu + b$ , for  $b$  tangible. Now the resultant matrix has entries  $1^\nu$  instead of  $1$ , so  $|\mathfrak{R}(f, g)|$  equals

$$\alpha_0^\nu + \alpha_1^\nu b + \alpha_2^\nu b^2 + \dots + \alpha_{m-1}^\nu b^{m-1} + \alpha_m b^m,$$

which is a ghost iff the  $\nu$ -value of  $b$  is at most that of  $\frac{\alpha_{m-1}}{\alpha_m}$ , the root of  $f$  with greatest  $\nu$ -value. Again, the resultant is a ghost iff  $f$  and  $g$  have a common tangible root.

**Example 4.8.** Suppose  $f = \lambda^2 + a^\nu \lambda + b$  and  $g = \lambda^2 + c^\nu \lambda + d$  are quadratic quasi-essential polynomials over a supertropical semifield; i.e.,

$$(4.2) \quad \nu(a^2) \geq b^\nu \quad \text{and} \quad \nu(c^2) \geq d^\nu.$$

Accordingly

$$(4.3) \quad Z_{\tan}(f) = \{x \in \mathcal{T} \mid (b/a)^\nu \leq x^\nu \leq a^\nu\} \quad \text{and} \quad Z_{\tan}(g) = \{x \in \mathcal{T} \mid (d/c)^\nu \leq x^\nu \leq c^\nu\}.$$

The resultant  $|\mathfrak{R}(f, g)|$  of  $f$  and  $g$  is given by:

$$|\mathfrak{R}(f, g)| = \begin{vmatrix} b & a^\nu & \mathbb{1} \\ & b & a^\nu \\ d & c^\nu & \mathbb{1} \\ & d & c^\nu \end{vmatrix} = d^2 + (acd)^\nu + (bd)^\nu + (bc^2)^\nu + (a^2d)^\nu + (abc)^\nu + b^2,$$

whose essential part, by (4.2), is

$$(4.4) \quad d^2 + (acd)^\nu + (bc^2)^\nu + (a^2d)^\nu + (abc)^\nu + b^2 = bf(c^\nu) + dg(a^\nu).$$

We show that  $f$  and  $g$  have no common tangible roots iff  $|\mathfrak{R}(f, g)| \in \mathcal{T}$ , in which case obviously  $|\mathfrak{R}(f, g)| = b^2 + d^2$ .

( $\Rightarrow$ ) Suppose  $Z_{\tan}(f) \cap Z_{\tan}(g) = \emptyset$ . Thus,  $b^\nu > (ac)^\nu$  or  $d^\nu > (ac)^\nu$ ; by symmetry, we may assume the first case, that  $b^\nu > (ac)^\nu$ . Then

$$(a^2c)^\nu > (bc)^\nu > (ac^2)^\nu > (ad)^\nu,$$

yielding  $(ac)^\nu > d^\nu$  and thus

$$(b^2)^\nu > (a^2c^2)^\nu > (a^2d)^\nu.$$

Also  $a^\nu \geq (\frac{b}{a})^\nu > c^\nu$  implies

$$(b^2)^\nu > (abc)^\nu > (bc^2)^\nu;$$

finally,

$$(a^2d)^\nu > (bd)^\nu > (acd)^\nu > (d^2)^\nu,$$

yielding altogether  $|\mathfrak{R}(f, g)| = b^2$ .

( $\Leftarrow$ ) Suppose that  $|\mathfrak{R}(f, g)|$  is tangible; then  $|\mathfrak{R}(f, g)| = b^2$  or  $|\mathfrak{R}(f, g)| = d^2$ . Assuming the former, we have  $(b^2)^\nu > (abc)^\nu$ ; i.e.,  $b^\nu > (ac)^\nu$ , implying  $Z_{\tan}(f) \cap Z_{\tan}(g) = \emptyset$ .

For intuition and future reference, we claim that the  $\nu$ -value of (4.4) equals that of

$$(4.5) \quad (a + c) \left( a + \frac{d}{c} \right) \left( \frac{b}{a} + c \right) \left( \frac{b}{a} + \frac{d}{c} \right).$$

Indeed, by symmetry we may assume that  $a^\nu \geq c^\nu$ . But (4.5) has the same  $\nu$ -value as

$$\begin{aligned} f(c)f\left(\frac{d}{c}\right) &= (c^2 + a^\nu c + b) \left( \left(\frac{d}{c}\right)^2 + a^\nu \frac{d}{c} + b \right) \\ &= d^2 + a^\nu cd + bc^2 + \frac{a^\nu d^2}{c} + (a^2)^\nu d + a^\nu cb + b\left(\frac{d}{c}\right)^2 + a^\nu b \frac{d}{c} + b^2, \end{aligned}$$

which matches (4.4) except for the extra terms  $\frac{a^\nu d^2}{c}$ ,  $b(\frac{d}{c})^2$ , and  $a^\nu b \frac{d}{c}$ , which are dominated respectively by  $d^2$ ,  $bd$ , and  $a^\nu bc$ ;  $bd$  is dominated in turn by  $b^2 + d^2$ .

Although the formula for the supertropical determinant is somewhat formidable, and is quite intricate even for quadratic polynomials, it becomes much simpler when the resultant is tangible, so our strategy is to reduce computations of the resultant to the tangible case as quickly as possible.

These examples indicate that the resultant is a ghost iff the polynomials  $f$  and  $g$  have a common root. The proof of this result involves an inductive argument, which we prepare with some notation. Given a polynomial  $f = \sum_{i=0}^m \alpha_i \lambda^i$ , we define

$$f_{[\ell]} = \sum_{i=\ell}^m \alpha_i \lambda^{i-\ell}, \quad \ell = 1, \dots, m;$$

thus,  $f = \lambda f_{[1]} + \alpha_0 = \lambda^2 f_{[2]} + \alpha_1 \lambda + \alpha_0 = \dots$ . Recall from [3, Lemma 7.28] that when  $\alpha_1$  is tangible, the polynomial  $f = \sum_{i=0}^m \alpha_i \lambda^i$  can be factored as  $(\lambda + \frac{\alpha_0}{\alpha_1}) f_{[1]}$ .

**Lemma 4.9.** *If  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \sum_{j=0}^n \beta_j \lambda^j$  are full polynomials, then*

$$|\mathfrak{R}(f, g)| = \alpha_0 |\mathfrak{R}(f, g_{[1]})| + \beta_0 |\mathfrak{R}(f_{[1]}, g)|.$$

*Proof.* We expand the resultant

$$|\mathfrak{R}(f, g)| = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & & & \\ & \alpha_0 & \alpha_1 & \alpha_2 & \dots & & \\ & & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \\ & & & \ddots & \dots & \ddots & \vdots \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n & & \\ & \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n & \\ & & \beta_0 & \beta_1 & \beta_2 & \dots & \vdots \\ & & & \ddots & \dots & & \end{vmatrix}$$

along the first column, to get

$$(4.6) \quad \alpha_0 \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & & \\ & \alpha_0 & \alpha_1 & \alpha_2 & \dots & \\ & & \ddots & \dots & \ddots & \dots \\ \beta_1 & \beta_2 & \dots & \beta_n & & \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_n & \\ & \ddots & \dots & \ddots & & \ddots \end{vmatrix} + \beta_0 \begin{vmatrix} \alpha_1 & \alpha_2 & \dots & & \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \\ & \ddots & \ddots & \dots & \dots \\ \beta_0 & \beta_1 & \dots & \beta_n & \dots \\ & \beta_0 & \beta_1 & \dots & \beta_n \\ & & \ddots & \dots & \ddots \end{vmatrix}$$

In computing the second supertropical determinant of (4.6) by expanding along the first column, the occurrence of  $\alpha_0$  in the second row must be multiplied by some  $\alpha_j$  in the first row, whereas, switching the first two rows, we also have  $\alpha_1 \alpha_{j-1}$ , which has greater  $\nu$ -value than  $\alpha_0 \alpha_j$  since  $f$  is essential. (Strictly speaking, we must also consider the possibility that  $(\alpha_0 \alpha_j)^\nu = (\alpha_1 \alpha_{j-1})^\nu$ , but in this case  $\alpha_1$  must be ghost, so again the term with  $\alpha_0 \alpha_j$  is not relevant to the computation of the supertropical determinant.) Thus the occurrence of  $\alpha_0$  in the second row cannot contribute to the second supertropical determinant of Equation (4.6), and we may erase it. Likewise, each occurrence of  $\alpha_0$  does not contribute to the second supertropical determinant of Equation (4.6).

By the same token, each occurrence of  $\beta_0$  does not contribute to the first supertropical determinant of Equation (4.6). Thus, (4.6) equals

$$\begin{aligned} & \alpha_0 \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & & \\ & \alpha_0 & \alpha_1 & & & \dots \\ & & \ddots & \dots & & \\ \beta_1 & \beta_2 & \dots & \beta_n & & \\ & \beta_1 & \beta_2 & \dots & \beta_n & \\ & & \ddots & \dots & & \end{vmatrix} + \beta_0 \begin{vmatrix} \alpha_1 & \alpha_2 & \dots & & \\ \alpha_1 & \alpha_2 & \dots & & \\ & \ddots & \dots & & \\ \beta_0 & \beta_1 & \dots & \beta_n & \\ & \beta_0 & \beta_1 & \dots & \\ & & \ddots & \dots & \end{vmatrix} \\ &= \alpha_0 |\mathfrak{R}(f, g_{[1]})| + \beta_0 |\mathfrak{R}(f_{[1]}, g)|. \end{aligned}$$

□

**Lemma 4.10.** *If  $f = \sum_{i=0}^m \alpha_i \lambda^i$  is a full polynomial and  $g = \lambda^2 + \beta_1 \lambda + \beta_0$  is irreducible with  $(\beta_1^2)^\nu \geq \beta_0^\nu$ , then*

$$|\mathfrak{R}(f, g)| = \sum_{\ell=0}^{m-1} \alpha_\ell \beta_0^\ell f_{[\ell]}(\beta_1) + \beta_0^m g(\alpha_{m-1}).$$

*Proof.* By definition  $g_{[1]} = \lambda + \beta_1$ , and thus  $|\Re(f_{[\ell]}, g_{[1]})| = f_{[\ell]}(\beta_1)$  for each  $\ell = 0, \dots, m-1$ ; cf. Example 4.8 and 4.7. Use Lemma 4.9 recursively to write

$$\begin{aligned}
|\Re(f, g)| &= \alpha_0 |\Re(f, g_{[1]})| + \beta_0 |\Re(f_{[1]}, g)| \\
&= \alpha_0 f(\beta_1) + \alpha_1 \beta_0 |\Re(f_{[1]}, g_{[1]})| + \beta_0^2 |\Re(f_{[2]}, g)| \\
&= \alpha_0 f(\beta_1) + \alpha_1 \beta_0 f_{[1]}(\beta_1) + \beta_0^2 |\Re(f_{[2]}, g)| \\
&= \alpha_0 f(\beta_1) + \alpha_1 \beta_0 f_{[1]}(\beta_1) + \alpha_2 \beta_0^2 |\Re(f_{[2]}, g_{[1]})| + \beta_0^3 |\Re(f_{[3]}, g)| \\
&= \dots \\
&= \sum_{\ell=0}^{m-1} \alpha_\ell \beta_0^\ell f_{[\ell]}(\beta_1) + \beta_0^m |\Re(f_{[m-1]}, g)| \\
&= \sum_{\ell=0}^{m-1} \alpha_\ell \beta_0^\ell f_{[\ell]}(\beta_1) + \beta_0^m g(\alpha_{m-1}).
\end{aligned}$$

□

**Remark 4.11.** We quote [3, Proposition 7.28]: Suppose  $f = \sum_j \alpha_j \lambda^j \in F[\lambda]$  is full. If  $\alpha_i \lambda^i$  is a tangible essential monomial of  $f$ , then

$$(4.7) \quad f = (\alpha_t \lambda^{t-i} + \alpha_{t-1} \lambda^{t-i-1} + \dots + \alpha_{i+1} \lambda + \alpha_i) \left( \lambda^i + \frac{\alpha_{i-1}}{\alpha_i} \lambda^{i-1} + \dots + \frac{\alpha_0}{\alpha_i} \right).$$

In the notation of this paper,

$$(4.8) \quad f = \left( \lambda^i + \frac{\alpha_{i-1}}{\alpha_i} \lambda^{i-1} + \dots + \frac{\alpha_0}{\alpha_i} \right) f_{[i]}.$$

We are ready for a formula for the resultant. It is convenient to start with the tangible case, both because it is more straightforward and also it helps in tackling the general case.

**Theorem 4.12.** Suppose that  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \sum_{j=0}^n \beta_j \lambda^j$  are both full polynomials over a supertropical semifield  $F$ , where the  $\alpha_i, \beta_j \neq 0_F$ .

(i) If all the  $\alpha_i, \beta_j$  are tangible, and  $a_i = \frac{\alpha_{i-1}}{\alpha_i}$  and  $b_i = \frac{\beta_{i-1}}{\beta_i}$ , then

$$(4.9) \quad |\Re(f, g)| = \alpha_m^n \beta_n^m \prod_{i=1}^m \prod_{j=1}^n (a_i + b_j) = \alpha_m^n \beta_n^m \prod_{i,j} |\Re(\lambda + a_i, \lambda + b_j)|.$$

(ii) In general, take tangible  $a_i$  such that  $(\alpha_i a_i)^\nu = (\alpha_{i-1})^\nu$  and  $b_j$  such that  $(\beta_j b_j)^\nu = (\beta_{j-1})^\nu$ , for  $0 \leq i < m$ ,  $0 \leq j < n$ . Then

$$|\Re(f, g)|^\nu = \alpha_m^n \beta_n^m \prod_{i=1}^m \prod_{j=1}^n (a_i + b_j)^\nu.$$

(iii) Notation as in (ii) and Lemma 4.9, if  $a_1^\nu > b_1^\nu$ , then

$$(4.10) \quad |\Re(f, g)| = \alpha_0 |\Re(f, g_{[1]})|.$$

(iv) For any polynomials  $f, g$ , and  $h$ ,

$$(4.11) \quad |\Re(f, gh)|^\nu = |\Re(f, g)|^\nu |\Re(f, h)|^\nu \quad \text{and} \quad |\Re(fg, h)|^\nu = |\Re(f, h)|^\nu |\Re(g, h)|^\nu.$$

*Proof.* (i) Noting that  $g_{[1]} = \sum_{j=1}^n \beta_j \lambda^{j-1}$  and  $b_1 = \frac{\beta_0}{\beta_1}$ , we have

$$g = g_{[1]} h$$

where  $h = \lambda + b_1$ ; in particular  $\beta_0 = \beta_1 b_1$ . Likewise, we have  $f = (\lambda + a_1) f_{[1]}$ . Also Lemma 4.9 yields

$$(4.12) \quad |\Re(f, g)| = \alpha_0 |\Re(f, g_{[1]})| + \beta_0 |\Re(f_{[1]}, g)|.$$

By hypothesis that  $f$  is full, we have  $a_1^\nu \leq a_2^\nu \leq a_3^\nu \leq \dots$ .

Our strategy is to consider the remaining cases:

- $a_1^\nu \neq b_1^\nu$  in which case we want to show that one of the terms on the right side dominates the other, and equals  $|\Re(f, g_{[1]})| |\Re(f, h)|$  (and thus also equals  $|\Re(f, g)|$  by bipotence).

- $a_1^\nu = b_1^\nu$ , in which case we want to show that both of the terms on the right side of Equation (4.12) have  $\nu$ -value equal to  $|\mathfrak{R}(f, g_{[1]})|^\nu |\mathfrak{R}(f, h)|^\nu$ , whereby  $|\mathfrak{R}(f, g)|$  is ghost and equal to  $|\mathfrak{R}(f, g_{[1]})| |\mathfrak{R}(f, h)|$ .

If  $a_1^\nu > b_1^\nu$ , then  $b_1^\nu < a_1^\nu \leq a_i^\nu$ , implying  $(b_1^i)^\nu < (a_i b_1^{i-1})^\nu$ , and thus  $(\alpha_i b_1^i)^\nu < (\alpha_{i-1} b_1^{i-1})^\nu < \dots \leq (\alpha_1 b_1)^\nu < (\alpha_0)^\nu$ . Thus, by bipotence,  $\alpha_0 = f(b_1) = |\mathfrak{R}(f, h)|$ , so the first term of the right side of (4.12) is

$$\alpha_0 |\mathfrak{R}(f, g_{[1]})| = |\mathfrak{R}(f, h)| |\mathfrak{R}(f, g_{[1]})|,$$

which equals the right side of Equation (4.9) by induction. Hence, to prove  $|\mathfrak{R}(f, g)| = |\mathfrak{R}(f, g_{[1]})| |\mathfrak{R}(f, h)|$ , we need only show that  $\beta_0 |\mathfrak{R}(f_{[1]}, g)|$  has  $\nu$ -value  $< |\mathfrak{R}(f, g_{[1]})| |\mathfrak{R}(f, h)|$ . (This also proves (iii) for tangible polynomials.) By induction on  $m$ ,  $|\mathfrak{R}(f_{[1]}, g)| = |\mathfrak{R}(f_{[1]}, g_{[1]})| |\mathfrak{R}(f_{[1]}, h)|$ . By Lemma 4.9,  $\beta_1 |\mathfrak{R}(f_{[1]}, g_{[1]})|$  has  $\nu$ -value  $\leq |\mathfrak{R}(f_{[1]}, g)|^\nu$ . But

$$b_1 |\mathfrak{R}(f_{[1]}, h)|^\nu = b_1 (f_{[1]}(b_1))^\nu < f(b_1)^\nu = |\mathfrak{R}(f, h)|^\nu,$$

so multiplying together (noting that  $\beta_0 = \beta_1 b_1$ ), we see that

$$\beta_0 |\mathfrak{R}(f_{[1]}, g)|^\nu = \beta_1 |\mathfrak{R}(f_{[1]}, g_{[1]})| b_1 |\mathfrak{R}(f_{[1]}, h)|^\nu < |\mathfrak{R}(f, g_{[1]})| |\mathfrak{R}(f, h)|^\nu,$$

as desired.

We want to conclude that

$$(4.13) \quad |\mathfrak{R}(f, g)| = \alpha_m^n \beta_n^m \prod_{i=1}^m (a_i + b_1) \prod_{j=2}^n (a_i + b_j).$$

Note that  $|\mathfrak{R}(f, h)| = f(b_1) = \alpha_m \prod (a_i + b_1)$ , whereas, by induction,

$$|\mathfrak{R}(f, g_{[1]})| = \alpha_m^{n-1} \beta_n^m \prod_{i=1}^m \prod_{j=2}^n (a_i + b_j);$$

we get (4.13) by multiplying these together.

If  $a_1^\nu < b_1^\nu$ , then  $b_1 = h(a_1)$ , and thus the second term of the right side of (4.12) is

$$\beta_0 |\mathfrak{R}(f_{[1]}, g)| = h(a_1) \beta_1 |\mathfrak{R}(f_{[1]}, g)|,$$

and we get (4.13) by the same induction argument (applied now to the left side).

Finally, if  $a_1^\nu = b_1^\nu$ , then the same argument shows that the two terms on the right side of Equation (4.12) are both  $\nu$ -matched to the right side of Equation (4.9), implying that the right side of Equation (4.12) is ghost, and it remains to show that the left side is also ghost. But this is clear since the assumption  $a_1^\nu = b_1^\nu$  implies that  $a_1$  is a common root of  $f$  and  $g$ . Thus, we have verified (4.13), yielding (i); we also have obtained (iii) for tangible polynomials.

(ii) and (iii) follow, since we can replace the  $\alpha_i$  and  $\beta_j$  by tangible coefficients of the same  $\nu$ -value.

(iv) follows for the same reason, since once we replace the coefficients of  $f$ ,  $g$ , and  $h$  by tangible coefficients of the same  $\nu$ -value, we may factor them further and apply (i).  $\square$

**Corollary 4.13.** *Suppose  $f = \prod_{i=1}^m (\lambda + a_i)$  and  $g = \prod_{j=1}^n (\lambda + b_j)$  are tangible. Then*

$$|\mathfrak{R}(f, g)| = \prod_{i,j} (a_i + b_j) = \prod_j f(b_j) = \prod_i g(a_i).$$

We turn to full polynomials over a supertropical semifield  $F$  (with  $\mathcal{T} = \mathcal{T}(F)$ ), recalling their decomposition from [3, Section 7].

**Definition 4.14.** *A full polynomial  $f = \sum_{i=0}^t \alpha_i \lambda^i$  is **semitangibly-full** if  $\alpha_t$  and  $\alpha_0$  are tangible, but  $\alpha_i$  are ghost for all  $0 < i < t$ ;  $f$  is **left semitangibly-full**, (resp. **right semitangibly-full**) if  $\alpha_0$  is tangible and  $\alpha_i$  are ghost for all  $0 < i \leq t$  (resp.  $\alpha_t$  is tangible and  $\alpha_i$  are ghost for all  $0 \leq i < t$ ).*



**Remark 4.15.** Suppose  $f = \sum_{i=0}^t \alpha_i \lambda^i$  is a full polynomial. Then, taking  $a_1 = \frac{\hat{\alpha}_0}{\alpha_1} \in \mathcal{T}$  and  $a_t = \frac{\hat{\alpha}_{t-1}}{\alpha_t} \in \mathcal{T}$ , we have (in logarithmic notation)

$$(4.14) \quad Z_{\text{tan}}(f) = \begin{cases} [a_1, a_t] & \text{for } f \text{ semitangibly-full;} \\ [a_1, \infty) & \text{for } f \text{ left semitangibly-full;} \\ (-\infty, a_t] & \text{for } f \text{ right semitangibly-full,} \end{cases}$$

as seen by inspection. Indeed, the  $\nu$ -smallest tangible root of  $f$  is  $a_1$  when  $\alpha_0$  is tangible (since for any  $a$  of  $\nu$ -value  $< a_1'$ , one has  $f(a) = \alpha_0$ ). Likewise, the  $\nu$ -largest tangible root of  $f$  is  $a_t$  when  $\alpha_t$  is tangible.

Put in the terminology of Definition 3.7, if  $f$  is left semitangibly-full, then  $f$  is  $a_1'$ -left half-tangible; if  $f$  is right semitangibly-full, then  $f$  is  $a_t'$ -right half-tangible.

Equation (4.7) shows that we can always factor  $f$  at tangible essential monomials into factors with disjoint root sets, leading immediately to the following assertion [3, Proposition 7.36]:

**Proposition 4.16.** Any full polynomial  $f$  can be decomposed as a product

$$(4.15) \quad f = f^{\text{l.s.}} f_1 \cdots f_t f^{\text{r.s.}}$$

where the polynomial  $f^{\text{l.s.}}$  is semitangibly-full or left semitangibly-full,  $f_1, \dots, f_t$ , are semitangibly-full, and  $f^{\text{r.s.}}$  is semitangibly-full or right semitangibly-full polynomial, and their tangible root sets are mutually disjoint intervals with descending  $\nu$ -values.

**Remark 4.17.** Equation (4.8) implies that in this decomposition (4.15),

$$f_{[i]} = f^{\text{l.s.}} f_1 \cdots f_t f_{[i]}^{\text{r.s.}}$$

for each  $i \leq \deg(f^{\text{r.s.}})$ .

Thus it makes sense for us to compute the resultant of semitangibly-full polynomials.

**Lemma 4.18.** Suppose that  $f = \sum_{i=0}^m \alpha_i \lambda^i$ ,  $g$ , and  $h = \sum_{j=0}^n \beta_j \lambda^j$  ( $n \geq 1$ ) are polynomials whose root sets are disjoint intervals; assume that all roots of  $f$  and  $g$  have  $\nu$ -value greater than every tangible root of  $h$ . Then

$$|\mathcal{R}(f, gh)| = \alpha_0^n \beta_n^m |\mathcal{R}(f, g)|,$$

which is tangible iff  $|\mathcal{R}(f, g)|$  is tangible. Explicitly, for each  $i \leq n$ ,

$$|\mathcal{R}(f, gh)| = \alpha_0^i |\mathcal{R}(f, (gh)_{[i]})|.$$

*Proof.* If  $n = 1$  then the assertion is clear from Example 4.7(i), so we assume that  $n > 1$ . By Theorem 4.12(iii),

$$|\mathcal{R}(f, gh)| = \alpha_0 |\mathcal{R}(f, (gh)_{[1]})|.$$

But the same argument shows that  $|\mathcal{R}(f, (gh)_{[1]})| = \alpha_0 |\mathcal{R}(f, (gh)_{[2]})|$ , and we have

$$|\mathcal{R}(f, gh)| = \alpha_0^2 |\mathcal{R}(f, (gh)_{[2]})|.$$

Iterating, after  $i$  steps we get

$$|\mathcal{R}(f, gh)| = \alpha_0^i |\mathcal{R}(f, (gh)_{[i]})|;$$

taking  $i = n$  yields

$$|\mathcal{R}(f, gh)| = \alpha_0^n \beta_n^m |\mathcal{R}(f, g)|,$$

as desired.  $\square$

Note that in Lemma 4.18,  $f$  could be either left semitangibly-full or semitangibly-full, and  $h$  could be either semitangibly-full or right semitangibly-full, but in every case the result is the same.

**Theorem 4.19.** Suppose that  $f = \sum_{i=0}^m \alpha_i \lambda^i$  and  $g = \sum_{j=0}^n \beta_j \lambda^j$  are both full polynomials over a supertropical semifield  $F$ , where the  $\alpha_i, \beta_j \neq 0_F$ .

- (i) Suppose  $f$  and  $g$  are full polynomials, decomposed as products as in Proposition 4.16; i.e.,

$$f = f^{1.s.} f_1 \cdots f_t f^{r.s.}, \quad g = g^{1.s.} g_1 \cdots g_u g^{r.s.},$$

and suppose the tangible root sets of  $f$  and  $g$  are disjoint. Let  $f_0 = f^{1.s.}$ ,  $g_0 = g^{1.s.}$ ,  $f_{t+1} = f^{r.s.}$ , and  $g_{u+1} = g^{r.s.}$ . Then

$$|\mathfrak{R}(f, g)| = \prod_{j=0}^{t+1} \prod_{k=0}^{u+1} |\mathfrak{R}(f_j, g_k)|,$$

each of which can be calculated according to Lemma 4.18.

- (ii) Specifically, if  $|\mathfrak{R}(f, g)|$  is tangible and  $g = \prod_j (\lambda + b_j)$ , then  $|\mathfrak{R}(f, g)| = \prod_j f(b_j)$ .
- (iii)  $|\mathfrak{R}(f, g)|$  is ghost iff  $f$  and  $g$  have a common root.

*Proof.*

- (i) We may assume that every tangible root of  $g^{r.s.}$  has  $\nu$ -value less than every tangible root of  $f$  as well as every tangible root of  $g^{1.s.} g_1 \cdots g_u$ . Let  $n_u = \deg(g^{r.s.})$ . Remark 4.17 applied to Lemma 4.18 implies

$$\begin{aligned} |\mathfrak{R}(f, g)| &= \alpha_0^i |\mathfrak{R}(f, g_{[n_u]})| = \alpha_0^i |\mathfrak{R}(f, g^{1.s.} g_1 \cdots g_u g^{r.s.}_{[n_u]})| \\ &= \alpha_0^{n_u} \beta_n^m |\mathfrak{R}(f, g^{1.s.} g_1 \cdots g_u)| \\ &= |\mathfrak{R}(f, g^{r.s.})| |\mathfrak{R}(f, g^{1.s.} g_1 \cdots g_u)|, \end{aligned}$$

and one continues by induction on  $t + u$ .

- (ii) Follows from (i).
- (iii) Follows from Remark 4.6 and the contra positive of (i). If  $|\mathfrak{R}(f, g)|$  is ghost, and the root sets are disjoint, some  $|\mathfrak{R}(f_j, g_k)|$  must be ghost, contradicting Lemma 4.18. □

Putting together Theorems 3.14 and 4.19 yield our main result:

**Theorem 4.20.** *Polynomials  $f = \sum \alpha_i \lambda^i$  and  $g = \sum \beta_j \lambda^j$  satisfy  $|\mathfrak{R}(f, g)| \in \mathcal{G}_0$  iff  $f$  and  $g$  are not relatively prime, iff  $f$  and  $g$  have a common tangible root.*

The next example illustrates the assertion of Theorem 4.20.

**Example 4.21.** *Let*

$$f = (\lambda + a)(\lambda + b) = \lambda^2 + b\lambda + ab \quad \text{and} \quad g = \lambda + c,$$

where  $a, b, c \in R$  and  $b^\nu > a^\nu$ . Then

$$(4.16) \quad \mathfrak{R}(f, g) = \begin{vmatrix} ab & b & 1 \\ c & 1 & \\ c & 1 & \end{vmatrix} = c^2 + bc + ab.$$

- (1) If  $f$  and  $g$  have a common tangible root, that is  $a$  or  $b$  respectively, then  $c^\nu = a^\nu$  (resp.  $c^\nu = b^\nu$ ), and clearly  $\mathfrak{R}(f, g) = (ab)^\nu$  (resp.  $\mathfrak{R}(f, g) = (b^2)^\nu$ ) is ghost.
- (2) When  $c^\nu \neq a^\nu, b^\nu$ , and thus  $f$  and  $g$  have no common factor, and  $\mathfrak{R}(f, g) \in \mathcal{G}_0$ , then at least one term in (4.16) is ghost. As usual, we take tangible  $\hat{a}, \hat{b}, \hat{c}$  such that  $(\hat{a})^\nu = a^\nu$ ,  $(\hat{b})^\nu = b^\nu$ , and  $(\hat{c})^\nu = c^\nu$ .

Assume first that  $\mathfrak{R}(f, g) = (ab)^\nu$  and  $a$  or  $b$  is ghost. Then  $c^\nu \leq a^\nu$  and thus  $\hat{c}$  is also a root of  $f$ .

If  $\mathfrak{R}(f, g) = (bc)^\nu$ , then  $a^\nu < c^\nu < b^\nu$ . If  $b$  is ghost, then  $\hat{c}$  is a common root of  $f$  and  $g$ . But if  $b$  is tangible, then  $c$  is ghost and  $\hat{a}$  is a common root of  $f$  and  $g$ .

Finally, if  $\mathfrak{R}(f, g) = c^2$ , where  $c$  is ghost, then  $a^\nu, b^\nu \leq c^\nu$ , so  $\hat{a}$  and  $\hat{b}$  are common tangible roots of  $f$  and  $g$ .

**4.1. A second proof of Corollary 4.13 using the generic method.** Since Corollary 4.13 encapsulates the basic property of the resultant, let us present a second proof using a different approach of independent interest. We start with a multivariate version of [3, Lemma 7.6].

**Lemma 4.22.** *Suppose  $f \in F[\lambda_1, \dots, \lambda_n]$ , and let  $f^a(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n)$  denote the specialization of  $f$  under  $\lambda_k \mapsto a \in \mathcal{T}$ . Suppose that the polynomial  $f^a$  becomes ghost on a nonempty open interval  $W_a$  of  $F^{(n-1)}$  and also assume that every tangible open interval  $W_{\mathcal{T}}$  of  $F^{(n)}$  contains some point  $\mathbf{b}$  that  $f(\mathbf{b}) \notin \mathcal{G}_0$ . Then  $(\lambda_k + a)$   $e$ -divides  $f$ .*

*Proof.* By symmetry of notation, one may assume  $k = n$ .

The case  $n = 1$  is just [3, Lemma 7.6], since the assertion is that  $f \in F[\lambda_1]$  becomes a constant that is a ghost on an open interval, and thus is a ghost; i.e.,  $a$  is a root of  $f$  which by assumption is ordinary. Thus, we may assume  $n > 1$ . Write  $f = \sum_i f_i \lambda_1^i$ , for  $f_i \in F[\lambda_2, \dots, \lambda_n]$ . In particular,  $f^a = \sum_i f_i^a \lambda_1^i$ . But the  $f_i^a \lambda_1^i$  are  $\nu$ -distinct on some open interval since they involve different powers of  $\lambda_1$ . Hence, by Lemma 1.10, each  $f_i^a \lambda_1^i$  (and thus  $f_i$ ) is ghost on some nonempty open interval, so by induction  $(\lambda_n + a)$   $e$ -divides each  $f_i$ , and thus also  $e$ -divides  $f$ .  $\square$

*Second proof of Corollary 4.13:* Using the generic method, we consider the case where all the  $a_i$  are (tangible) indeterminates over the supertropical semifield  $F$ ; then  $|\mathfrak{R}(f, g)|$  is some polynomial in  $F[a_1, \dots, a_m]$ . But, substituting  $a_i \mapsto b_j$  yields a common root for  $f$  and  $g$ , and thus sends  $|\mathfrak{R}(f, g)|$  to  $\mathcal{G}_0$ , in view of Theorem 4.20. Hence, by Lemma 4.22,  $(a_i + b_j)$   $e$ -divides  $|\mathfrak{R}(f, g)|$  for each  $i, j$ , and these are all distinct, implying by an easy induction argument that  $\prod_{i,j} (a_i + b_j)$   $e$ -divides  $|\mathfrak{R}(f, g)|$ .

Clearly,  $\prod_{i,j} (a_i + b_j)$  has degree  $mn$ . So let us compute the degree of  $|\mathfrak{R}(f, g)|$ . For  $i \leq n$ , the  $(i, j)$  term in  $\mathfrak{R}(f, g)$  (when nonzero) has degree  $m + j - i$ . For  $i > n$ , the  $(i, j)$  term in  $\mathfrak{R}(f, g)$  (when nonzero) has degree 0. Thus it follows from the formula for calculating the supertropical determinant that  $|\mathfrak{R}(f, g)|$  has degree  $mn + \sum j - \sum i = mn$ . One concludes  $|\mathfrak{R}(f, g)| = c \prod_{i,j} (a_i + b_j)$  for some  $c \in F$ . But the term in  $\prod_{i,j} (a_i + b_j)$  without any  $b_j$  is precisely  $(a_1 \dots a_m)^n$ , which occurred by itself in  $|\mathfrak{R}(f, g)|$ . Thus  $c = \mathbb{1}_F$ , proving the first assertion. The other equalities follow at once. For example,  $\prod_i (b_j + a_i) = f(b_j)$ , so

$$\prod_{i,j} (a_i + b_j) = \prod_j f(b_j).$$

$\square$

## 5. BÉZOUT'S THEOREMS

One of the major applications of the resultant to geometry is Bézout's theorem. Throughout this section we assume that  $F$  is a supertropical semifield. Suppose  $f, g$  in  $F[\lambda_1, \lambda_2]$ . Rewriting the polynomials in terms of  $\tilde{\lambda} = \lambda_1/\lambda_2$  and  $\lambda = \lambda_2$ , the polynomials  $f$  and  $g$  can be viewed as polynomials in  $\lambda$ , with coefficients in  $F[\tilde{\lambda}]$ . From this point of view, the resultant  $|\mathfrak{R}(f, g)|$  is a polynomial  $p(\tilde{\lambda})$ .

**Theorem 5.1 (Bézout's theorem).** *Nonconstant polynomials  $f, g$  in  $F[\lambda_1, \lambda_2]$  cannot have more than  $mn$  2-ordinary points in the intersection of their sets of projective roots, where  $m = \deg(f)$  and  $n = \deg(g)$ .*

*Proof.* Assume that the tangible points  $(x_i, y_i)$  lie on each root set,  $C_f$  and  $C_g$ , defined respectively by the roots of  $f$  and  $g$ , for  $i = 1, \dots, mn + 1$ . After a suitable additive translation, cf. Remark 2.1, we may assume that each  $y_i \neq 0_F$ . Then, after a suitable Frobenius morphism (Remark 2.4), we may assume that the  $\frac{x_i}{y_i}$  are distinct (as well as finite). Let  $\tilde{\lambda} = \frac{\lambda_1}{\lambda_2}$ , and view  $f, g$  as polynomials in  $R[\lambda_2]$ , where  $R = F[\tilde{\lambda}]$ .

Viewing  $|\mathfrak{R}(f, g)|$  in  $R = F[\tilde{\lambda}]$ , one sees that for any specialization  $\tilde{f}$  and  $\tilde{g}$  given by  $\tilde{\lambda} \mapsto x_i/y_i$ ,  $\tilde{f}$  and  $\tilde{g}$  have the common 2-ordinary root  $y_i$ , and thus their resultant is ghost. In other words  $(\tilde{\lambda} + x_i/y_i)$   $e$ -divides  $|\mathfrak{R}(f, g)|$  for each  $i = 1, \dots, mn + 1$ . Hence  $\deg(\mathfrak{R}(f, g)) > mn + 1$ . But by definition  $|\mathfrak{R}(f, g)|$  has degree  $mn$  – a contradiction.  $\square$

## 6. SUPERTROPICAL DIVISIBILITY AND THE NULLSTELLENSATZ

In [3, Theorem 6.16] we proved a Nullstellensatz involving tangible polynomials. In order to formulate a more comprehensive version, we need a more general notion of divisibility. Suppose  $R$  is a semiring with ghosts, which satisfies supertropicality (Note 1.1).

**Definition 6.1.** An element  $g \in R$  **supertropically divides**  $f \in R$  if  $f + qg$  is ghost with  $(f + qg)^\nu = f^\nu$ , for a suitable  $q \in \mathcal{T}$ .

**Example 6.2.** Let  $R = F[\lambda]$ , and consider the polynomial  $f = \lambda^2 + 6^\nu \lambda + 7$ , whose tangible root set is the interval  $[1, 6]$ .

(i) If  $g = \lambda + 4$ , whose tangible root set is  $\{4\}$ , then

$$f + (\lambda + 3)g = f + \lambda^2 + 4\lambda + 7$$

is ghost.

(ii)  $g = \lambda^2 + 4^\nu \lambda + 6$ , whose tangible root set is the interval  $[2, 4]$ , then

$$f^2 + (\lambda^2 + 8)g = (\lambda^4 + 6^\nu \lambda^3 + 12^\nu \lambda^2 + 13^\nu \lambda + 14) + (\lambda^4 + 4^\nu \lambda^3 + 8\lambda^2 + 12^\nu \lambda + 14),$$

which is ghost.

**Example 6.3.** Suppose  $f = \lambda^2 + a_2^\nu \lambda + a_1 a_2$ , for  $a_1, a_2$  tangible. Then, for a tangible,  $\lambda + a$  supertropically divides  $f$  iff  $a_1^\nu \leq a^\nu \leq a_2^\nu$ . Indeed, for the constant term of  $f + (\lambda + a)q$  to be ghost, we must have  $(aq)^\nu = (a_1 a_2)^\nu$ . The coefficient of  $\lambda$  shows that  $\max\{q^\nu, a^\nu\} < a_2^\nu$ , and thus  $\min\{q^\nu, a^\nu\} < a_1^\nu$ .

**Definition 6.4.** Suppose  $A \subset R$ . The **supertropical radical**  ${}^{\text{trop}}\sqrt{A}$  is defined as the set

$$\{a \in R : \text{some power } a^k \text{ supertropically divides an element of } A\},$$

which in other words is

$$\{a \in R : (a^k + b)^\nu = (a^k)^\nu \text{ and } a^k + b \in \mathcal{G}_0, \text{ for some } b \in A \text{ and some } k \in \mathbb{N}^+\}.$$

An ideal  $A$  of  $R$  is **supertropically radical** if  $A = {}^{\text{trop}}\sqrt{A}$ .

**Remark 6.5.** If  $A$  is an ideal of a commutative semiring  $R$ , then  ${}^{\text{trop}}\sqrt{A} \triangleleft R$ . Indeed, if  $a_1^{k_1} + b_1 \in \mathcal{G}_0$  and  $a_2^{k_2} + b_2 \in \mathcal{G}_0$ , then by the Frobenius map,

$$(a_1 + a_2)^{k_1 k_2} + b_1^{k_2} + b_2^{k_1} = a_1^{k_1 k_2} + b_1^{k_2} + a_2^{k_1 k_2} + b_2^{k_1} = (a_1^{k_1} + b_1)^{k_2} + (a_2^{k_2} + b_2)^{k_1}$$

which is ghost, of the same  $\nu$ -value as  $a_1^{k_1 k_2} + a_2^{k_1 k_2} = (a_1 + a_2)^{k_1 k_2}$ . Likewise, for all  $r$  in  $R$ ,  $(ra_1)^{k_1} + r^{k_1} b_1 \in \mathcal{G}_0$ , of the same  $\nu$ -value as  $(ra_1)^{k_1}$ .

By the same sort of argument, if  $R$  is a commutative supertropical semiring and  $A$  is a sub-semiring of  $\text{Fun}(R^{(n)}, R)$ , then  ${}^{\text{trop}}\sqrt{A}$  is also a sub-semiring of  $\text{Fun}(R^{(n)}, R)$ .

**6.1. The comprehensive supertropical Nullstellensatz.** The comprehensive supertropical version of the Hilbert Nullstellensatz is as follows (with the same proof as in [3, Theorem 6.16]).

For a polynomial  $f \in F[\lambda_1, \dots, \lambda_n]$ , we define the set

$$D_f = \{\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{T}_0^{(n)} : f(\mathbf{a}) \in \mathcal{T}\};$$

thus  $\mathcal{T}_0^{(n)} \setminus D_f$  is the set of tangible roots of  $f$  in  $\mathcal{T}_0^{(n)}$ . Refining this definition, writing  $f = \sum f_{\mathbf{i}}$ , a sum of monomials,  $D_{f, \mathbf{i}}$  is defined to be

$$D_{f, \mathbf{i}} = \{\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{T}_0^{(n)} : f(\mathbf{a}) = f_{\mathbf{i}}(\mathbf{a}) \in \mathcal{T}\}.$$

Therefore,  $\mathbf{a} \in D_{f, \mathbf{i}}$  iff  $f$  has the monomial  $f_{\mathbf{i}}$  with  $f_{\mathbf{i}}(\mathbf{a})$  tangible, and  $f_{\mathbf{i}}(\mathbf{a})^\nu > f_{\mathbf{j}}(\mathbf{a})^\nu$  for all  $\mathbf{j} \neq \mathbf{i}$ ; hence  $D_f = \bigcup_{\mathbf{i}} D_{f, \mathbf{i}}$ .

Clearly, the  $D_{f, \mathbf{i}}$  are open sets, and each has a finite number of connected components  $D_{f, \mathbf{i}_u}$ , for  $1 \leq u \leq t = t_{\mathbf{i}}$ , called the **irreducible components** of  $f$ . Note that  $D_f$  is the disjoint union of the  $D_{f, \mathbf{i}_u}$ .

Given an irreducible component  $D$  of  $f$ , we write  $f \preceq_D g$  if  $g$  has an irreducible component containing  $D$ ;  $f \in_{\text{ir-com}} S$  for  $S \subseteq F[\lambda_1, \dots, \lambda_n]$ , if for every irreducible component  $D = D_{f, i_u}$  of  $f$  there is some  $g \in S$  (depending on  $D_{f, i_u}$ ) with  $f \preceq_D g$ .

We recall [3, Definition 6.12].

**Definition 6.6.** *Given an irreducible component  $D$  of  $f$ , we write  $f \preceq_D g$  if  $g$  has an irreducible component containing  $D$ ;  $f \in_{\text{ir-com}} S$  for  $S \subseteq F[\Lambda]$ , if for every irreducible component  $D = D_{f, i_u}$  of  $f$  there is some  $g \in S$  (depending on  $D_{f, i_u}$ ) with  $f \preceq_D g$ .*

*Also, define the **dominant monomial** of  $f$  on the irreducible component  $D$ , denoted  $f_D$ , to be that monomial  $f_i$  such that  $f(\mathbf{a}) = f_i(\mathbf{a}) \in \mathcal{T}$  for every  $\mathbf{a} \in D$ .*

**Theorem 6.7. (Comprehensive supertropical Nullstellensatz)** *Suppose  $F$  is a connected,  $\mathbb{N}$ -divisible, supertropical semifield,  $A \triangleleft F[\lambda_1, \dots, \lambda_n]$ , and  $f \in F[\lambda_1, \dots, \lambda_n]$ . Then  $f \in_{\text{ir-com}} A$  iff  $f \in {}^{\text{trop}}\sqrt{A}$ .*

*Proof.* We review the proof in [3, Theorem 6.16]. Again, the direction  $(\Leftarrow)$  is clear, so we prove  $(\Rightarrow)$ . Let  $\hat{f}$  denote the tangible polynomial having the same  $\nu$ -value as  $f$ . Write  $D_{\hat{f}}$  as the disjoint union of the irreducible components  $D_{\hat{f}, i}$  of the complement set of  $Z_{\text{tan}}(\hat{f})$ , which we number as  $D_1, \dots, D_q$ . Some of these remain as components of the complement set of  $Z_{\text{tan}}(f)$ ; we call these components “true”. Other components are roots of  $f$  (because of its extra ghost coefficients) and thus belong to  $Z_{\text{tan}}(f)$ ; and we call these components “fictitious.” For each true component  $D_k$  take a polynomial  $g_k \in A$  with an irreducible component  $D'$  containing  $D_k$ . Let  $g_{k, \mathbf{j}} = \beta_{\mathbf{j}} \lambda_1^{j_1} \cdots \lambda_n^{j_n}$ ,  $\mathbf{j} = (j_1, \dots, j_n)$ , be the dominant monomial of  $g_k$  on  $D'$ .

At any stage, we may replace  $f$  by a power  $f^m$  (and, if necessary,  $g_k$  by  $g_k^m$  times some element of  $\mathcal{T}$ ), for this does not affect its irreducible components; we do this where convenient.

As in the proof in [3, Theorem 6.16], on each true  $D_k$ ,  $f_k = g_{k, \mathbf{j}}$ . But [3, Lemma 6.14], and induction on the number of (possibly fictitious) components separating  $D_k$  from the other components still shows that  $f^{m'}$  dominates  $g_k$  on each component. Now we apply the same argument to all true neighbors, and continue until we have taken into account all of the (finitely many) true components. The proof is then completed by taking  $m > \max_k \{m_k + m'_k\}$ ; then  $f^m = \sum_k g_k \in A$  on the true components and  $f^m$  dominates  $\sum_k g_k$  on the fictitious components, implying  $(f^m)^\nu = (f^m + \sum_k g_k)^\nu$  and  $f^m + \sum_k g_k$  is ghost, as desired.  $\square$

**6.2. Bézout’s Theorem revisited.** Theorem 5.1 probably can be generalized to the non-tangible situation, but we do not yet have a full proof:

**Conjecture 6.8 (Supertropical Bézout conjecture).** *Nonconstant polynomials  $f, g$  in  $F[\lambda_1, \lambda_2]$  cannot have more than  $mn$  tangible connected components in the intersection of their sets of tangible roots, where  $m = \deg(f)$  and  $n = \deg(g)$ .*

Here is a proposed method to prove this conjecture. First, we need a more comprehensive version of Lemma 4.22.

**Lemma 6.9.** *Suppose  $f \in F[\lambda_1, \dots, \lambda_n]$ , and  $W$  is a component of  $Z_{\text{tan}}(f)$ . If the projection of  $W$  on the  $k$  component is a closed interval  $[b_1, b_2]$  where  $b_1^\nu < b_2^\nu$ , then for any  $a$  with  $b_1^\nu \leq a^\nu \leq b_2^\nu$ , the polynomial  $(\lambda_k + a)$  supertropically divides  $f$ .*

*Proof.* The case  $n = 1$  follows from [3, Proposition 7.47], since the assertion is that for each element  $a$  in the interval  $[b_1, b_2]$ , the polynomial  $f$  specializes to a constant that is a ghost on a tangible open interval, and thus is a ghost; i.e.,  $a$  is a root of  $f$  in  $F[\lambda_1]$ , in view of Example 6.3. Thus, we may assume  $n > 1$ . Write  $f = \sum_i f_i \lambda_1^i$ , for  $f_i \in F[\lambda_2, \dots, \lambda_n]$ . In particular,  $f^a = \sum_i f_i^a \lambda_1^i$ . But the  $f_i^a \lambda_1^i$  are  $\nu$ -distinct on some open interval since they involve different powers of  $\lambda_1$ . Hence, by Lemma 1.10, each  $f_i^a \lambda_1^i$  (and thus  $f_i$ ) is ghost on some nonempty open interval, so by induction  $(\lambda_k + a)$  supertropically divides each  $f_i$ , and thus also supertropically divides  $f$ .  $\square$

The difficulty in completing the proof along the lines of the proof of Theorem 5.1 is that one cannot say for  $\lambda + a_i$  supertropically dividing  $f$  for  $1 \leq i \leq t$  that also  $\prod_j (\lambda + a_j)$  supertropically divides  $f$ . For example, this is false for  $f = \lambda^2 + 6^\nu \lambda + 7$ ,  $a_1 = 2$ ,  $a_2 = 3$ , and  $a_3 = 5$ . The reason is that these roots all

lie on the same connected component. Presumably, one may be able to complete the proof by counting the number of connected components of the complement set of the resultant, with respect to a suitable projection onto the line.

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